Loudspeaker crossover networks

By

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Abstract

Loudspeaker systems use crossover networks directing low and high frequencies to individual loudspeaker units optimised for limited frequency ranges. The introduction of a crossover network should not degrade the resultant performance but the loudspeakers are physically separated, which introduces problems around the crossover frequency when listening off-axis, and the individual responses of the loudspeaker units further complicates summation of the output signals. The front baffle introduces reflections from the edges and the listening room adds reflections from its boundaries causing interference with the direct signal seriously affecting the resultant response of the loudspeaker system.

The objective of this report is the study of crossover networks and the different causes that degrades the performance.

Contents

Foreword

The current project was initiated as a three-week course to be executed in August of the 2005 summer vacation since I could not participate in the normal three-week period in June. My professor *Finn Agerkvist* accepted the proposal of a project to study crossover networks.

The main objective was the design of crossover networks realising a transfer function of unity, i.e. flat amplitude and zero phase, and I planned to include the effect of loudspeaker bandwidth, the problems associated with off-axis listening due to the displacement of the loudspeaker on the front baffle and the interference from reflections within the listening room. Finn Agerkvist suggested that I also included the reflections due to diffraction.

Initially I planned to use SPICE for simulations and a spread sheet for calculations, but I soon realised that it was more appropriate to base the simulations and calculations on MATLAB.

I decided to work through the loudspeaker model presented by *Leach* in order to derive a useful model for loudspeakers, and I included the effect of the voice coil inductance and combined the low and high frequency models from *Leach* into one single model, which covers the frequency range below diaphragm break-up.

As the project progressed, I realised the need to include group delay and it seemed appropriate to add notes on the threshold of hearing thus defining an acceptance limit for use during the development of a crossover network.

According to my log, I have been working for 180 hours, which is 50 % more than the nominal workload for a three-week course. If an unlimited amount of time were available, I would have worked more on high-order crossover filters and improved the sections on diffraction, off-axis listening, boundary reflection and group delay.

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1. Introduction

Ideally, a single loudspeaker should reproduce the full audible frequency range without any detectable distortion, but this is unfortunately not possible although good full-range loudspeakers do exist. The frequency range of a full-range loudspeaker is limited with weak bass and unsatisfactory treble, the frequency response is irregular or at least compromised by the directivity at high frequencies and it is difficult to keep distortion low when the same diaphragm is used for bass and treble.

Low frequencies moves the diaphragm significantly at high sound pressure levels thus introducing harmonic distortion related to loudspeaker construction (magnet, voice coil and suspension) and inter-modulation between bas and treble caused by the Doppler-effect. The one and only way of distortion reduction is decreasing diaphragm excursion, but this require an increase of diaphragm area to compensate for the lost sound pressure; and enlarging loudspeaker size worsens high frequency reproduction.

It all boils down to a requirement of loudspeakers optimised for reproduction of a limited frequency range and thus the need of a frequency dividing network.

1.1. Crossover network

A pair of typical crossover networks are shown in Figure 1. To the left is a two-way system, which could use a crossover frequency around 2 kHz, and to the right is a three-way system, which could use crossover frequencies around 800 Hz and 4 kHz.

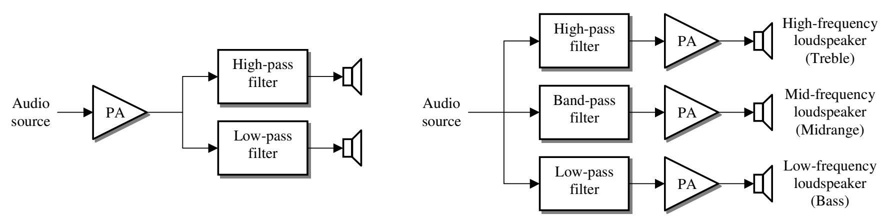


Figure 1 – Layout of a typical cross-over network, which can be passive, i.e. consisting of capacitors, inductors and resistors and driven from a single power amplifier (left), or it can be active, i.e. using operational amplifiers with frequency-dependent feedback and individual power amplifiers for each channel (right).

The loudspeakers are driven from power amplifiers, which can either be located before the crossover network; the conventional approach using passive crossover networks, or the power amplifier can be located between the crossover network and the loudspeaker; thus requiring an amplifier for each loudspeaker.

The passive crossover network is currently the most used approach but the active crossover network is expected to be increasingly popular in the near future since high-quality power amplifier modules based upon the switch-mode technique (Class D) are becoming a serious alternative to the linear power amplifiers of today. In addition to improved control of filter parameters and protection of the loudspeakers do the active crossover network offer electrical control of the moving system parameters and adjustment of the response to the listening room.

But before entering the study of crossover networks, a few words on what can be heard, and what cannot, is required.

1.2. Threshold of hearing

There is no idea in optimising a filter if the improvement is inaudible and the money could be spend better on other jobs. So, here is a brief overview of what is audible and what is not. Don't take the limits too literately; they are meant as guidelines.

1.2.1. Audible range

The audible range is defined by the Fletcher-Munson curves reproduced to the left in Figure 2. They published their data in 1933 using headphones; measurements using anechoic chambers were published in 1956 by Robinson-Dadson and later reviewed and standardised in 2003 as ISO 226, shown to the right.

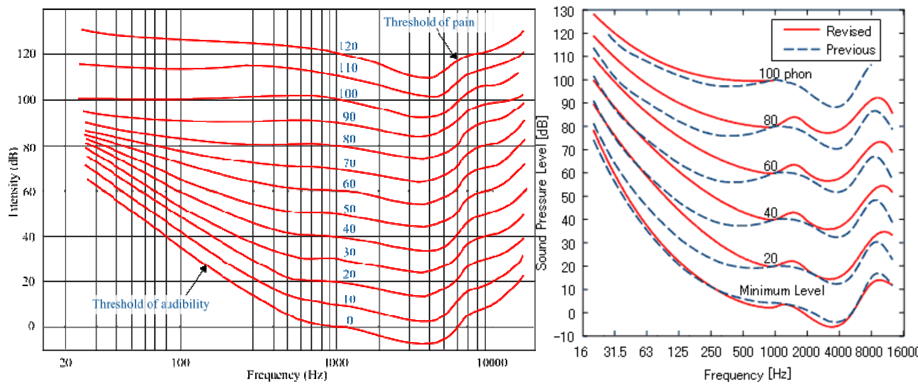


Figure 2 – Equal loudness contours due to Fletcher-Munson (http://en.wikipedia.org) and to Robinson-Dadson (http://www.aist.go.jp/aist_e/latest_research).

We can hear signals from 15 Hz to 15 kHz but few loudspeaker systems can reproduce this range, at least not at realistic levels since low-frequency signals require large diaphragms and long voice coils in order to move the air; the sound pressure level must exceed 80 dB at very low frequencies to be heard and 110 dB is required at 20 Hz to balance a typical speech level around 65 dB.

Organ music may extend to 16 Hz for organs fitted with 32 feet pipes, but they are rarely found – and rarely used – so organs music is limited to 32 Hz. Piano music may extend to 27 Hz, while jazz, rock and popular music seldom passes below 41 Hz; the lowest string on the acoustical double bass and the electric bass guitar. Powerful low-frequency signals may, however, arise from electrical keyboards, computerized effects and recordings of large drums, machines, thunder storms and explosions.

No one can hear sound above 20 kHz, and aging further reduces the limit, so a pragmatic upper limit would be 15 kHz. People with "golden ears" may postulate that the treble unit should extend far beyond 20 kHz to avoid phase distortion. The audibility of phase is controversial, so a safe view would be that reproduction beyond 20 kHz do not harm; but have in mind that FM-radio broadcasting limits the range to 15 kHz and CD-recordings to 20 kHz. Modern signal transmission using MPEG and other formats often use the 44.1 kHz sampling frequency of the CD-media thus sharing the limit.

1.2.2. Change of level

The ability to detect a change of level is between 0.5 dB and 2 dB [1] so loudspeaker artefacts below this limit can be expected inaudible. This is quite fortunate, since irregularities of this order of magnitude must be expected with loudspeakers. It is obvious that the lowest limit should be used designing high-end equipment, since the listener can be expected to have trained ears.

1.2.3. Group delay

Threshold of audibility of group delay is being debated, but it should be relatively safe stating the limit as a couple of milliseconds for signals in the 500 Hz to 8 kHz range. (Source: http://www.trueaudio.com/post_010.htm).

1.3. Musical instruments

The fundamental note of musical instruments is typical located below 1 kHz as shown in Figure 3. The harmonic overtones extend beyond the hearing limit but the level is reduced toward the higher frequencies. The decay is strongly dependent upon the actual instrument being examined, but an average slope would be around –6 dB/octave. Most music use fundamentals within the 3½ octave range from the lower C-note at 65 Hz to the upper g2-note at 780 Hz, so the musical power is mainly restricted to this range.

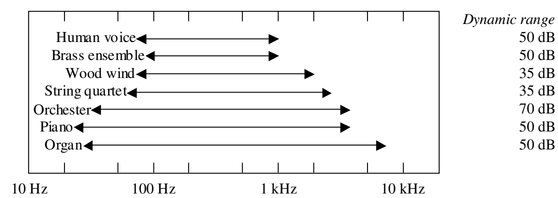


Figure 3 – Frequency range for the fundamental note of musical instruments [5].

The dynamic range extend from 40 dB to 110 dB for acoustical instruments and higher for electrically amplified music. A symphony or rock orchestra cannot be reproduced at realistic levels for home entertainment, so the playback level must typically be reduced.

Recording level was previously compressed manually by increasing the weakest levels during recording, in order to cut the master disk (LP records) keeping the weakest signals above the noise floor of the medium. This is not needed nowadays for recording of compact disks (CD), where the dynamic level is, at least theoretically, 96 dB.

Reducing the reproduction level moves the weakest signals below the threshold of hearing, at least for the lowest frequencies, so the loudspeaker system may require a bass boost for reproduction at reduced levels. This is sometimes called *physiological loudness contour* and is included with many amplifier systems. The success of this correction is dependent upon the set-up of the combined system, consisting of the source, the amplifier, the loudspeaker and the size of the room and is most effectively implemented with integrated systems where the interconnection levels are known.

The correction is often accompanied by a treble boost as well, but this is based on a misinterpretation of the equal loudness contour; the treble part of the equal loudness contour is turned upward at high frequency thus indicating reduced sensitivity at high frequencies, but the distance between the different levels is almost constant so correction is not required.

1.4. Cut-off slope

A crossover network is not a *brick wall filter* with infinite attenuation outside the pass band; the crossover network gradually attenuates the signal above or below the crossover frequency as illustrated in Figure 4 for low-pass and high-pass filters.

The *pass band* is the frequency range where attenuation is minimum, typically –3 dB although some filters attenuate more than this. The *transition band* is the frequency range where the attenuation is becoming active and the *stop band* is the frequency range where the attenuation has become sufficient to efficiently remove the loudspeaker.

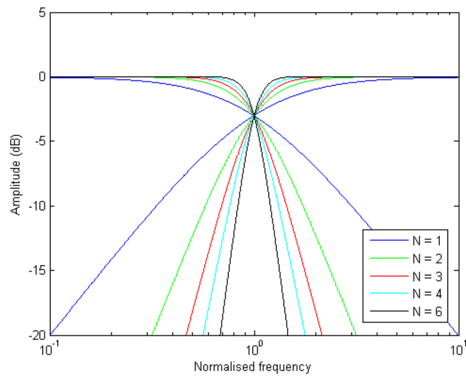


Figure 4 – Filter characteristics for Butterworth filters with orders 1, 2, 3, 4 and 6.

A loudspeaker contributes with audible output in the transition band, which must be taken into account to avoid interference between the loudspeakers. Sufficient amount of attenuation is obtained when the level from the attenuated channel is less than a certain limit, for instance below –20 dB. The limit defines the acceptable interference level.

> Assume for example, that the loudspeaker peaks 6 dB at some frequency near the stop band and that this must be attenuated. With –20 dB of intended attenuation this corresponds an actual level of –14 dB, or a sound pressure of $10^{-14/20}$ = 20 % of the nominal level, which may result in ±2 dB of interference.

A crossover network with a cut-off slope of ±6 dB/octave indicates that the loudspeaker must be well-behaved for at least 3 octaves beyond the cross-over frequency. A bass loudspeaker, which is to be cut-out above 2 kHz, must be reasonably flat to 16 kHz.

The crossover network must protect the treble loudspeaker from the high power levels at lower frequencies. The loudspeaker is compliance-controlled below the resonance frequency, which is usually around 1 kHz, so the diaphragm moves in proportion to the applied voltage. The low-frequency excursion of the diaphragm may be inaudible but it may give rise to audible distortion of high-frequency signals when the low-frequency excursion of the suspension system reaches the non-linear region. The unnecessary dissipation of power heats the voice coil and may damage the treble loudspeaker, which is capable of handling few watts only.

Most of the signal power in music is located below approximately 500 Hz so a power reduction well below 1 W of dissipation within the treble loudspeaker requires in excess of 20 dB of attenuation for sufficient protection. If the treble loudspeaker is to be cut-in at 2 kHz, which is just two octaves above 500 Hz, the required filter slope becomes a minimum of 12 dB/octave.

1.5. Transfer function

All channels of the crossover network will be described by their *transfer function*, which is a convenient way of describing an electrical filter, and system analogies allow straightforward transformation to mechanical and acoustical systems as well. The transfer function may be defined from requirements such as flat amplitude, which can then be used to specify the system parameters.

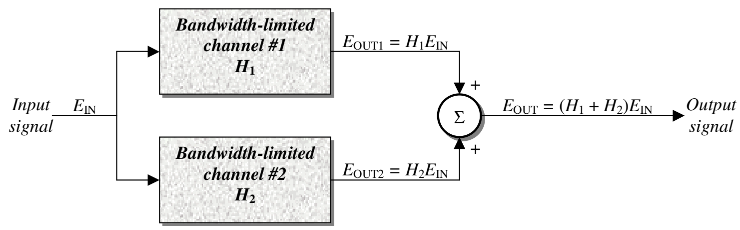


Figure 5 – A crossover network consists of two or more filter channels dividing the frequency range between the loudspeakers. The output from the loudspeakers are summed at the observation point; i.e. at the ear of the listener.

Transfer functions will be expressed in the *frequency domain* where frequency is represented by the *Laplace* operator s, defined by: $s = \alpha + i\omega$. The initial conditions are described by α and $\omega = 2\pi f$ is the angular frequency. Throughout this report is $\alpha = 0$, so $s = i\omega$ can be assumed although the formulas are generally valid and may, if required, be transformed to the time domain using the inverse Laplace transform. However, time domain representations are not referenced in this document.

The complex frequency operator will be normalised by division with ω_0, which represents the cut-off frequency in most situations.

$$s_0 = \frac{s}{\omega_0}$$

A transfer function H is defined by the excitation input to the network, E_{IN}, and the output response from the network, E_{OUT}. Assuming sinusoidal excitation:

$$E_{OUT} = HE_{IN} = HE_0 \exp(i\omega t)$$

An input excitation signal E_{IN} is routed in parallel to the channels of a network with the individual transfer functions H_1, H_2, ... and the output signals become:

$$E_{OUT1} = H_1E_{IN}, \quad E_{OUT2} = H_2E_{IN}, \quad \ldots$$

The loudspeakers will, for the moment, be considered ideal, so the acoustical output of the loudspeaker is a true copy of the electrical input signal. Assuming linearity, the signals from the individual channels are added at the receiver:

$$E_{OUT} = E_{OUT1} + E_{OUT2} + \ldots$$

Using the above definition of the transfer function, the sum can be written:

$$E_{OUT} = (H_1 + H_2 + \ldots + H_M)E_{IN}$$

The sum of the individual transfer functions is defined as the *system transfer function*.

$$H = H_1 + H_2 + \ldots H_M$$

The main objective of this report is analysing the system transfer function for the complete system involving the crossover network, the loudspeakers, the front baffle and the listening room. In order to do so, a reference transfer function is needed for comparison. It is obvious, that an ideal transfer function should not add or remove any information, it should be a string of wire.

$$H = 1$$

A scaling factor, different from unity, is allowed, since amplification, attenuation, sign inversion or time delay within the system is not considered a distortion of the signal. The scaling factor may also include a dimension for transformation between the electrical, mechanical or acoustical systems.

1.5.1. Ideal filters – Constant voltage filters

Although loudspeakers are far from ideal, avoiding approximations in the design of the crossover network it is a good starting point. The design of crossover networks fulfilling the requirement H = 1, which guarantees flat amplitude and a phase of zero, are called constant-voltage filters and will be based upon the following polynomial:

$$P_N = 1 + a_1 s_0 + a_2 s_0^2 + \cdots + a_{N-1} s_0^{N-1} + s_0^N$$

The constants a_1, a_2, …, a_{N-1} defines the filter type, N is the order of the polynomial and the coefficients are shown for the Butterworth filter type later in this chapter.

Other polynomials, such as Bessel or Chebychev, can use the above polynomial form if they are normalised to unity for coefficients a_0 and a_N. If this is not the case, all terms of the polynomials must be divided by a_0, and the normalisation coefficient ω_0 must thereafter be corrected to include a_0 and a_N.

An alternative representation of P_N is the product form, which is using the roots of the polynomial.

The third order Butterworth polynomial can be represented by the following two identical expressions:

$$P_3 = 1 + 2s_0 + 2s_0^2 + s_0^3$$
$$P_3 = (1 + s_0) \times (0.5 + i0.866 + s_0) \times (0.5 - i0.866 + s_0)$$

After the multiplications are carried out the original polynomial results.

A transfer function of order N can now be defined:

$$\boxed{H_N = \frac{Q_N}{P_N} = \frac{1 + b_1 s_0 + b_2 s_0^2 + \cdots + b_{N-1} s_0^{N-1} + s_o^N}{1 + a_1 s_0 + a_2 s_0^2 + \cdots + a_{N-1} s_0^{N-1} + s_o^N}}$$

The transfer function satisfy the requirement H = 1 when $a_i = b_i$ since the nominator polynomial Q_N and denominator polynomial P_N are identical, but the requirement will be violated, if $a_i \neq b_i$ for one or more of the terms. This violation may be required building crossover networks of high order where the ideal transfer function becomes cumbersome. The result is anyway a valid transfer function although the phase response and possibly also the amplitude response will be affected.

Division into crossover channels uses the following identity:

$$H_N = \frac{1}{P_N} + \frac{b_1 s_0}{P_N} + \frac{b_2 s_0^2}{P_N} + \cdots + \frac{b_{N-1} s_0^{N-1}}{P_N} + \frac{s_0^N}{P_N}$$

Each term represents a transfer function with its own characteristics and two or more terms can be combined as required. All terms share the same denominator polynomial and are thus of the same order, regardless of the actual order of the nominator.

Two examples will introduce the method.

Example 1. For a first-order crossover only two terms are available so the design leaves no choice other than accepting the following arrangement:

$$H_{LP1} = \frac{1}{P_N} = \frac{1}{1+s_0} \qquad H_{HP1} = \frac{s_0}{P_1} = \frac{s_0}{1+s_0}$$

Example 2. For a crossover network of sufficiently high order ($N \geq 4$), the first two terms of P_N can be used for the bass channel, the last two terms for the treble channel and the remaining terms are available for the midrange channel:

$$H_{LPN} = \frac{1 + b_1 s_0}{P_N} \qquad H_{BPN} = \frac{b_2 s_0^2 + \cdots + b_{N-2} s_0^{N-2}}{P_N} \qquad H_{HPN} = \frac{b_{N-1} s_0^{N-1} + s_0^N}{P_N}$$

The cut-off slopes are proportional to $1/f^{N-1}$ for the bass channel, f^{N-1} for the treble channel and f^2 and $1/f^2$ for the midrange channel. A fourth order filter would offer cut-off slopes of ±18 dB/octave for bass and treble and ±12 dB/octave for midrange.

Both filters are realised in the next chapter together with other implementations.

1.5.2. Non-ideal filters – All-pass filters

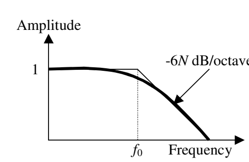
The above method is useful for filters up to fourth order but become cumbersome for higher orders. A solution is to remove all middle terms and use only the first and last terms representing the low-pass and high-pass channels. These filters, which are called all-pass filters, can be used for crossover networks of any order. The requirement $H = 1$ is not satisfied so the phase of the filter will be different from zero.

With $b_1, b_2, \ldots, b_{N-1}$ set to zero the transfer functions become:

$$H_{LPN} = \frac{1}{P_N} \quad and \quad H_{HPN} = \frac{s_0^N}{P_N} \quad \Rightarrow \quad H_N = \frac{1 + s_0^N}{P_N}$$

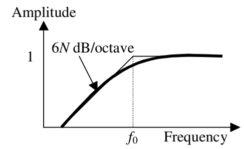
The first term represents a *low-pass channel* with a cut-off frequency of $f_0 = \omega_0/2\pi$ and a cut-off slope of $1/f^N$, so the higher frequencies are attenuated by $-6N$ dB/octave. Filter amplitude response is flat under certain conditions, which will be analysed below. First-order filters are ideal (constant-voltage filters) and will not need special considerations.

$$\begin{aligned} H_{LPN} &= \frac{1}{P_N} \xrightarrow[s_0 \to 0]{} 1 \\ &= \frac{1}{P_N} \xrightarrow[s_0 \to \infty]{} \frac{1}{s_0^N} \end{aligned}$$



The second term represents a *high-pass channel* with a cut-off frequency of $f_0 = \omega_0/2\pi$ and a cut-off slope of f^N, so the lower frequencies are attenuated by $6N$ dB/octave.

$$H_{HPN} = \frac{s_0^N}{P_N} \xrightarrow[s_0 \to 0]{} s_0^N$$
$$= \frac{s_0^N}{P_N} \xrightarrow[s_0 \to \infty]{} \frac{s_0^N}{s_0^N} = 1$$



When the output from both channels are combined the following transfer function for the complete system is obtained:

$$H_N = \frac{1+s_0^N}{P_N} \qquad \textit{At crossover}: \quad H_N = \frac{1+i^N}{P_N}$$

At crossover ($s_0 = i$) is the sum depending upon the filter order and for N = 2, 6, 10, ... the terms cancel – no signal is being transmitted at crossover. The cancellation can be avoided by sign inversion one of the channels, which has the consequence that the phase moves from 0° to –180° through the frequency range.

- For N odd are the channels 90° or 270° out of phase, and the channels combine at crossover with 3 dB of loss. To avoid peaking, the transfer functions must be designed to –3 dB at the crossover frequency and this can be realised using a Butterworth polynomial as basis for the design.
- For N even are two channels in-phase if the above mentioned sign inversion is included as required and the channels combine at crossover without loss. To avoid peaking, the transfer functions must be designed to –6 dB at the crossover frequency. This can be realised using a squared Butterworth polynomial as basis for the design; a design method referred to as the *Linkwitz-Riley* filter design.

Constant amplitude, $|H_N| = 1$, is realised by the so-called *all-pass filters*, which are based upon the Butterworth polynomials.

First-order crossover networks are born as ideal filters (i.e. constant voltage), so the amplitude is constant. This will be demonstrated below using variable $w = \omega/\omega_0$ as a substitute for s_0 in order to identify real and imaginary parts of the frequency variable. The all-pass variant of the first-order crossover network ($1 - s_0$ in the nominator) is also useful and is analysed as well [2].

$$|H_1| = \left|\frac{1 \pm s_0}{1+s_0}\right| = \left|\frac{1 \pm iw}{1+iw}\right| = \frac{\sqrt{1^2+w^2}}{\sqrt{1^2+w^2}} = 1$$

Filters of higher order can realise the all-pass function when the polynomial used is of Butterworth characteristic, since this allows factorisation of the nominator and denominator polynomials.

Second-order crossover networks require inversion of the high-pass channel to avoid the notch filter and the level must be –6 dB at crossover, so the crossover network is based on a squared first-order Butterworth polynomial. The amplitude response becomes [2]:

$$|H_2| = \left|\frac{1-s_0^2}{(1+s_0)^2}\right| = \left|\frac{(1+s_0)(1-s_0)}{(1+s_0)(1+s_0)}\right| = \left|\frac{1-s_0}{1+s_0}\right| = \left|\frac{1-iw}{1+iw}\right| = \frac{\sqrt{1^2+w^2}}{\sqrt{1^2+w^2}} = 1$$

Third-order crossover networks are insensitive to the sign of the treble channel (the amplitude response is unaltered although the phase response is changed). The third-order Butterworth polynomial use $a_1 = a_2 = 2$, and the factorisation results in [2]:

$$|H_3| = \left|\frac{1+s_0^3}{1+2s_0+2s_0^2+s_0^3}\right| = \left|\frac{(1+s_0)(1-s_0+s_0^2)}{(1+s_0)(1+s_0+s_0^2)}\right| = \left|\frac{1-iw-w^2}{1+iw-w^2}\right| = \frac{\sqrt{(1-w^2)^2+w^2}}{\sqrt{(1-w^2)+w^2}} = 1$$

$$|H_3| = \left|\frac{1-s_0^3}{1+2s_0+2s_0^2+s_0^3}\right| = \left|\frac{(1-s_0)(1+s_0+s_0^2)}{(1+s_0)(1+s_0+s_0^2)}\right| = \left|\frac{1-iw}{1+iw}\right| = \frac{\sqrt{1^2+w^2}}{\sqrt{1^2+w^2}} = 1$$

Fourth-order crossover networks require a level of –6 dB at crossover and can be based upon a squared second-order Butterworth polynomial ($a_1 = \sqrt{2}$). The amplitude response becomes [2]:

$$|H_4| = \left|\frac{1+s_0^4}{\left(1+\sqrt{2}s_0+s_0^2\right)^2}\right| = \left|\frac{\left(1+\sqrt{2}s_0+s_0^2\right)\left(1-\sqrt{2}s_0+s_0^2\right)}{\left(1+\sqrt{2}s_0+s_0^2\right)^2}\right| = \left|\frac{1-\sqrt{2}s_0+s_0^2}{1+\sqrt{2}s_0+s_0^2}\right|$$

$$= \left|\frac{1-i\sqrt{2}w+w^2}{1+i\sqrt{2}w+w^2}\right| = \frac{\sqrt{(1+w^2)^2}}{\sqrt{(1+w^2)^2}} = 1$$

Higher order filters are not covered in this report but can, according to reference [2], be shown to fulfil the all-pass filter requirement $|H| = 1$.

1.5.3. Butterworth filters

Crossover networks are often based upon the Butterworth polynomial since this leads to good all-pass filters. The amplitude response for a Butterworth low-pass filter is [5]:

$$|H_{LPN}| = \frac{1}{\sqrt{1+s_0^{2N}}} = \frac{1}{\sqrt{1+\left(\dfrac{\omega}{\omega_0}\right)^{2N}}} \xrightarrow[s_0\to\infty]{} \left(\frac{\omega_0}{\omega}\right)^N$$

$$\xrightarrow[s_0\to 0]{} 1$$

The amplitude approaches unity for zero frequency, the amplitude is $1/\sqrt{2}$ or –3 dB at the crossover frequency regardless of the filter order, and the asymptotic cut-off slope for higher frequencies is $-6N$ dB/octave. This is a maximally flat filter and one of its characteristics is, that all filter blocks of the physical implementation will be designed for the same cut-off frequency but with different quality factors.

Other filter characteristics are realised through frequency transformation, introduced briefly below [5], but the transformations are not required for the current analysis since the different channels are generated directly from the transfer function and not from transformation of a model filter.

A high-pass filter is realised by transformation of s_0 into $1/s_0$:

$$s_0 \rightarrow \frac{1}{s_0} \quad \Rightarrow \quad \left|H_{HPN}\right| = \frac{1}{\sqrt{1+\left(\dfrac{1}{s_0}\right)^{2N}}} = \frac{1}{\sqrt{1+\left(\dfrac{\omega_0}{\omega}\right)^{2N}}} \xrightarrow[s_0\to\infty]{} 1$$

$$\xrightarrow[s_0\to 0]{} \left(\frac{\omega}{\omega_0}\right)^N$$

A band-pass filter is realised by the following transformation, which consists of $1/s_0$ as well as s_0 in combination with a coefficient B, representing the bandwidth of the resulting filter:

$$s_0 \rightarrow \frac{1+s_0^2}{Bs_0} \quad \Rightarrow \quad \left|H_{BPN}\right| = \frac{1}{\sqrt{1+\left(\dfrac{1+s_0^2}{Bs_0}\right)^{2N}}} \xrightarrow[s_0\to\infty]{} \frac{1}{\left(\dfrac{s_0}{B}\right)^N} = B^N\left(\frac{\omega_0}{\omega}\right)^N$$

$$\xrightarrow[s_0=i]{} 1$$

$$\xrightarrow[s_0\to 0]{} \frac{1}{\left(\dfrac{1}{Bs_0}\right)^N} = B^N\left(\frac{\omega}{\omega_0}\right)^N$$

The coefficients and roots of the Butterworth polynomial is shown in the table below for filters from second to seventh order. The first-order polynomial $1 + s_0$ is Butterworth too but do not include coefficients to be defined.

Table 1 – Coefficients and roots of Butterworth polynomials (from [5]).

Order	Coefficients	Roots
2	a_1 = 1.4142	(-0.7071 ±i0.7071)
3	a_1 = a_2 = 2.0000	(-1) (-0.5000 ±i0.8660)
4	a_1 = a_3 = 2.6131 a_2 = 3.4142	(-0.3827 ±i0.9239) (-0.9239 ±i0.3817)
5	a_1 = a_4 = 3.2361 a_2 = a_3 = 5.2361	(-1) (-0.3090 ±i0.9511) (-0.8090 ±i0.5878)
6	a_1 = a_5 = 3.8637 a_2 = a_4 = 7.4641 a_3 = 9.1416	(-0.2588 ±i0.9659) (-0.7071 ±i0.7071) (-0.9659 ±i0.4339)
7	a_1 = a_6 = 4.4940 a_2 = a_5 = 10.0978 a_3 = a_4 = 14.5918	(-1) (-0.2225 ±i0.9749) (-0.6235 ±i0.7818) (-0.9010 ±i0.4339)

2. Crossover networks

Ideal filters, defined by the transfer function requirement $H = 1$, were expected to realise good crossover networks but this proved to be the case only for low-order networks; higher order networks tend to include terms complicating the construction and obstructing the operation. All-pass filters, realising two-way systems with maximum cut-off slope, were also analysed and proved to operate as expected.

2.1. First-order

Crossover networks of first order are popular because the component cost is low; it may sometimes be possible designing crossover networks with one single capacitor for the treble loudspeaker. However, the cut-off slope is ±6 dB/octave, which is insufficient in most cases since the loudspeakers must reproduce well 3 octaves past the crossover frequency. This is problematic since bass loudspeakers suffer from increased directivity and diaphragm break-up at high frequencies, which concentrates the sound on-axis and typically generates a ragged high-frequency response, and treble loudspeakers cannot reproduce below the resonance frequency.

In addition to this is the filter insufficient in protecting the treble loudspeaker against destructive low-frequency signals, so the first-order crossover is limited to loudspeakers intended for low sound pressure levels and is typically found in low-cost designs.

A first-order transfer function is defined from H_N with $N = 1$:

$$H_1 = \frac{1+s_0}{1+s_0}$$

2.1.1. Symmetrical – two way

The equation is separated into two channels with a low-pass channel for the bass loudspeaker and a high-pass channel for the treble loudspeaker.

$$\boxed{\begin{aligned} H_{HP1} &= \frac{s_0}{1+s_0} \\ H_{LP1} &= \frac{1}{1+s_0} \end{aligned}}$$

Realisation is simple, only one component is required for each branch but the filter is *very* dependent upon the impedance of the loudspeaker so the passive network may not prove satisfactory in real life.

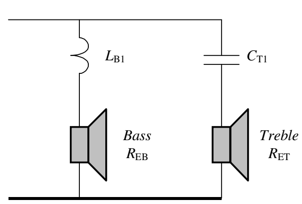


Figure 6 – Passive crossover network. The attenuation is as expected only when the impedance of the loudspeakers are constant, so impedance compensation is required in most cases – hence opposing the simplicity of the design.

The result is shown below. The channels add up to unity and the phase to zero, as they should since this is an ideal filter (constant-voltage) and assuming perfect loudspeakers with identical path lengths to the listener.

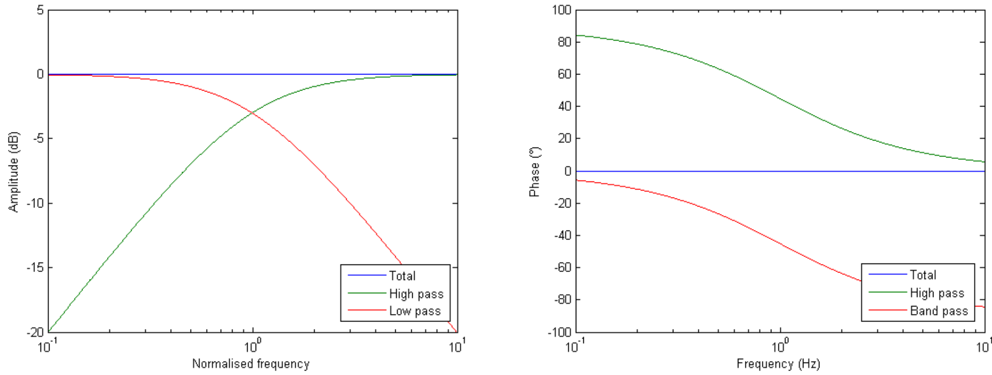


Figure 7 – Amplitude and phase response. The loudspeakers are 90° out of phase throughout frequency, thus adding with 3 dB of loss when combined. The channels are at –3 dB at crossover and add to 0 dB.

A variant of the filter is realised by inverting the treble loudspeaker, which generates the following transfer function:

$$H_1 = \frac{1 - s_0}{1 + s_0}$$

This is an all-pass filter, where the amplitude is flat but the phase is changing gradually from 0° to –180° through frequency with –90° at crossover.

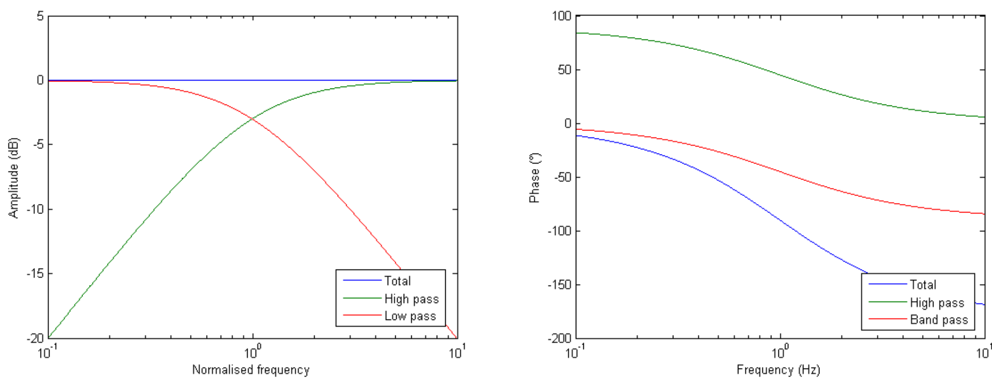


Figure 8 – Amplitude and phase response with the treble loudspeaker inverted. The loudspeakers are 90° out of phase throughout frequency.

The introduction of a phase different from zero for the complete crossover network introduces a group delay different from zero and this is shown in Figure 9.

Note that the group delay unit is calculated for a normalised filter corresponding to a cut-off frequency of 1 Hz. With a cut-off frequency of 1000 Hz the group delay scaling will be in milliseconds and not seconds.

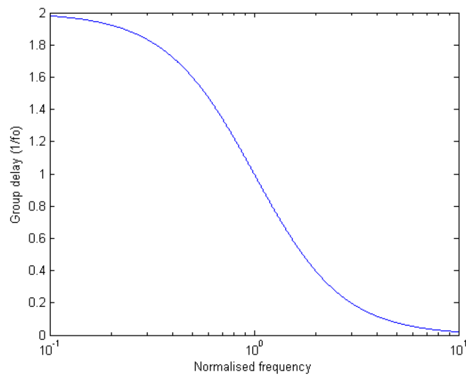


Figure 9 – Group delay with the treble loudspeaker inverted.

2.1.2. Using bass loudspeaker roll-off

Some bass loudspeakers are designed to roll off smoothly above a certain frequency, typically in the range where the crossover frequency is placed. The crossover network can be simplified if this roll off is used as the low-pass filter thus only implementing the high-pass filter capacitor for the treble loudspeaker. This results in a low-cost crossover network since only one capacitor is required. The bass loudspeaker must realise the required transfer function, so the –3 dB frequency of the bass loudspeaker dictates the crossover frequency.

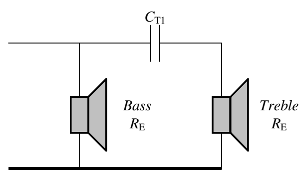


Figure 10 – Passive crossover network using the bass loudspeaker roll off.

Treble loudspeakers build from piezoelectric transducers with an integrated horn are available with a cut-off frequency of approximately 4 kHz and can be used without external components. The treble loudspeaker accepts up to 30 V applied directly and can be used for a two-way system without any components within the crossover network – but the cut-off is very sharp so the resulting design is not a first-order crossover network.

It is possible to electrically adjust the cut-off frequency of the bass loudspeaker using *pole-zero compensation*, which is most effectively implemented using active filtering. The method can be used to "move" the bass loudspeaker voice coil cut-off frequency from the actual value to the desired value.

Assume that the bass loudspeaker cut-off frequency is at ω_1, and not at ω_0 as required. The transfer function can be rewritten to include a null and a pole at the new frequency. The ratio of the new null/pole transfer function is unity so the transfer function is not changed.

$$H_{LP1} = \frac{1}{1+s_0} = \frac{1}{1+s_0} \times \frac{1+s_1}{1+s_1} = \frac{1+s_1}{(1+s_0)(1+s_1)} = H_{CN1} H_{LS1}$$

The terms are then arranged so the zero is moved to the crossover network H_{CN1}. The transfer functions for the crossover network H_{CN1} and the bass loudspeaker H_{LS1} then becomes:

$$H_{CN1} = \frac{1+s_1}{1+s_0} \qquad H_{LS1} = \frac{1}{1+s_1}$$

The crossover network H_{CN1} is now a correction filter, which modifies the amplitude spectrum of the low-frequency channel making the bass loudspeaker useful with the required crossover frequency. The correction should not be brought too far, however, but minor corrections of the order of ±6 dB (one octave up or down) should be realisable.

Note, that moving the crossover frequency upward requires amplification of the signal fed to the bass loudspeaker and moving the crossover frequency down requires attenuation of the signal fed to the bass loudspeaker. The former is impossible to implement using passive filters and the latter impractical, hence the recommendation of active filtering.

2.2. Second-order

Crossover networks of second order are popular due to the low component count and relatively steep cut-off slope. In addition is the designer provided with an interesting collection of filters to select among; both the ideal constant-voltage and the non-ideal all-pass filters are offered.

A second-order transfer function is defined from H_N with $N = 2$:

$$H_2 = \frac{1 + a_1 s_0 + s_0^2}{1 + a_1 s_0 + s_0^2}$$

Coefficient a_1 defines the filter characteristic around the crossover frequency and is conventionally defined by the quality factor Q of a second-order circuitry:

$$a_1 = \frac{1}{Q}$$

A common quality factor is 0.71 for the Butterworth characteristic, which is –3 dB at the crossover frequency, but any value can be used with $Q = 0.5$ to 1 as typical values.

2.2.1. Asymmetrical – two way

One obvious realisation of a two-way crossover network is to divide between the first-order and second-order terms thus increasing the cut-off slope for the treble loudspeaker to improve the protection.

$$\boxed{\begin{aligned} H_{HP2} &= \frac{s_0^2}{1 + a_1 s_0 + s_0^2} \\ H_{LP2} &= \frac{1 + a_1 s_0}{1 + a_1 s_0 + s_0^2} \end{aligned}}$$

The high-pass filter is of second-order with a slope of 12 dB/octave, which is sufficient to protect the treble loudspeaker. A useful range of 1.5 octaves below the cut-off frequency is required for 20 dB of attenuation so the crossover frequency must be higher than $2^{1.5} = 2.83$ times the resonance frequency of the treble loudspeaker.

At the crossover frequency, and assuming $a_1 = 2$, we get:

$$H_{HP2} = \frac{-1}{1 + 2i - 1} = \frac{-1}{2i} = \tfrac{1}{2} i = 0.5\angle 90°$$

$$H_{LP2} = \frac{1 + 2i}{1 + 2i - 1} = \frac{1 + 2i}{2i} = 1 - \tfrac{1}{2} i = 1.1\angle -27°$$

So, the channels are 117° apart and will add with some loss, hence the slight boost of the low-frequency channel at the crossover frequency. For $a_1 = 1$ both channels are boosted but the sum remains constant at unity so the phase of the channels are moved further apart for reduced value of a_1. This is not a good way of designing a crossover network; the channels should not oppose each other since the result then becomes a difference between two fighting channels. A sound way of designing is to select a fairly large value of the coefficient, where $a_1 = 2$ seems to be a fair compromise.

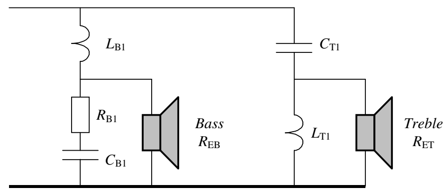


Figure 11 – Passive crossover network.

Active realisation:

$$H_{LP2} = 1 - H_{HP2}$$

The transfer function H_{HP2} is implemented as a standard second-order high-pass filter with $Q = 1/a_1$, and the low-pass channel can be derived by an operational amplifier. A value of $a_1 = 1$ results in $Q = 1$, which is a 1 dB Chebychev characteristic with relative steep cut-off and this also applies to the low-pass channel, which is cut-off after one decade. A value of $a_1 = 2$ results in $Q = 0.5$, which is the limit where the roots of the polynomial becomes real. This removes any tendency to oscillate in the treble channel, hence the smooth transition without peaking. The cut-off of the low-pass channel is somewhat weaker so the bass loudspeaker must operate well to at least twenty times the cut-off frequency.

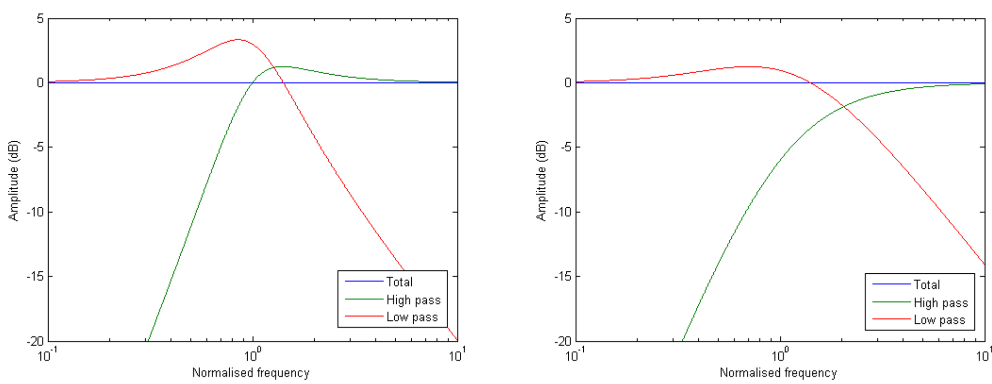


Figure 12 – Amplitude response with a_1 = 1 (left) and a_1 = 2 (right). The quality factor refers to the treble channel and is Q = 1 (left) and Q = 0.5 (right).

The low-pass filter is second-order but the first-order term in the nominator reduces the cut-off slope to –6 dB/octave at high frequencies, which requires a bass loudspeaker capable of operating 3 octaves above the cut-off frequency, so the channel is more or less full range and the bass loudspeaker must perform well at high frequencies.

2.2.2. Symmetrical – two way

The first-order term of the transfer function can be split into two halves so both channels includes a first-order term.

$$\boxed{\begin{aligned} H_{HP2} &= \frac{\frac{a_1}{2}s_0 + s_0^2}{1 + a_1 s_0 + s_0^2} \\ H_{LP2} &= \frac{1 + \frac{a_1}{2}s_0}{1 + a_1 s_0 + s_0^2} \end{aligned}}$$

Both filters are of second-order but the slope is ±6 dB/octave, which is insufficient to protect the treble loudspeaker so the solution should not be used for high-power systems. The loudspeakers must be capable of operating 3 octaves outside the crossover frequency

At the crossover frequency, and assuming $a_1 = 2$, we get:

$$\begin{aligned} H_{HP2} &= \frac{i-1}{1+2i-1} = \frac{i-1}{2i} = 0.71\angle 45° \\ H_{LP2} &= \frac{1+i}{1+2i-1} = \frac{1+i}{2i} = 0.71\angle -45° \end{aligned}$$

So, the channels are 90° apart at crossover and will add with 3 dB loss to 0 dB. Reducing the coefficient to $a_1 = 1$ introduces peaking in both channels to compensate for the increased phase difference.

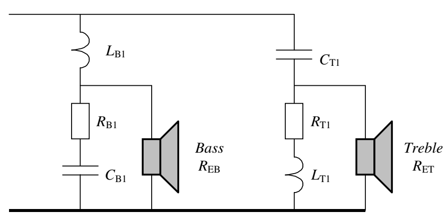


Figure 13 – Passive crossover network.

The coefficient adjusts the behaviour of the filter and two examples are shown below. The design with $a_1 = 1$ results in steep cut-off but there is a tendency for ringing on transients although the two channels will cancel when combined. It could be taken as a warning for problems with off-axis listening where the output from the bass loudspeaker is reduced due to its directivity.

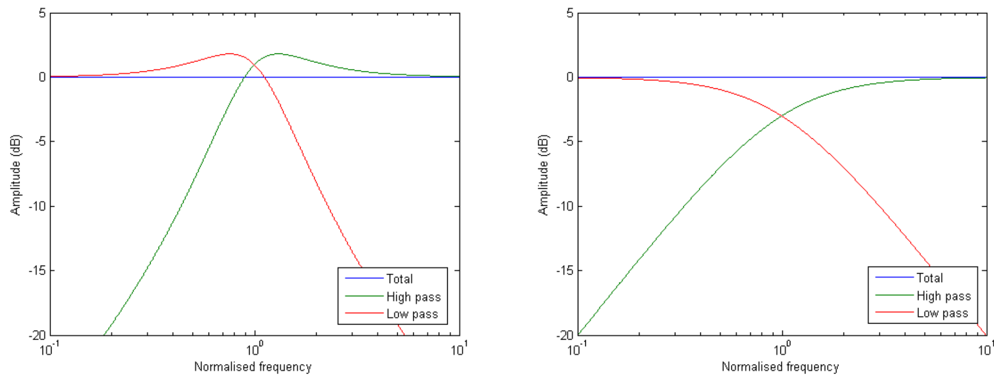


Figure 14 – Amplitude response with a_1 = 1 (left) and a_1 = 2 (right).

It appear that a_1 should be in the range from 0.5 to 1 as a starting point at least. A very smooth result is obtained with a value of 1.6, which is shown in Figure 15.

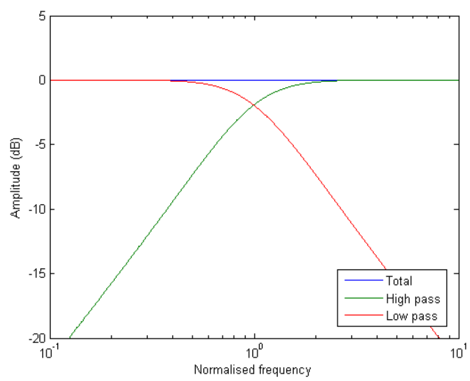


Figure 15 – Amplitude response with a_1 = 1.6.

2.2.3. Symmetrical – three way

The equation can be split into three channels:

$$\boxed{\begin{aligned} H_{HP2} &= \frac{s_0^2}{1 + a_1 s_0 + s_0^2} \\ H_{BP2} &= \frac{a_1 s_0}{1 + a_1 s_0 + s_0^2} \\ H_{LP2} &= \frac{1}{1 + a_1 s_0 + s_0^2} \end{aligned}}$$

The cut-off slope is ±12 dB/octave for the bass and treble channels and ±6 dB/octave for the midrange channel so the treble loudspeaker is protected but the midrange loudspeaker must cover a range of 6 octaves total. Coefficient a_1 represents the quality factor (Q = 1/ a_1), thus defining the pulse response of the individual channels; the resultant pulse response of the complete system is unity when the outputs are combined, assuming perfect addition of the channels.

At the crossover frequency, and assuming $a_1 = 2$, we get:

$$H_{HP2} = \frac{-1}{1+2i-1} = \frac{-1}{2i} = 0.50\angle 90°$$

$$H_{BP2} = \frac{2i}{1+2i-1} = 1\angle 0°$$

$$H_{LP2} = \frac{1}{1+2i-1} = \frac{1}{2i} = 0.50\angle -90°$$

So, the low-pass and high-pass channels cancel at crossover and leaves the midrange to fill the gap. The design was originally proposed by Bang & Olufsen and labelled as the *Filler Driver* system. The name indicates that the middle channel was not considered a conventional midrange channel but rather a phase correction of a two-way system.

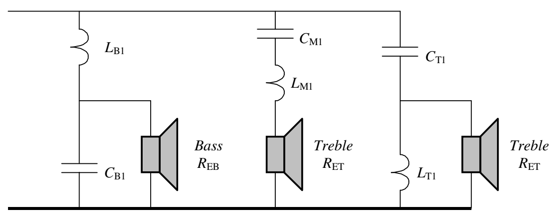


Figure 16 – Passive crossover network.

Two examples are shown, using Q = 1, which represents a 1 dB Chebychev filter characteristic of the second-order filter, and Q = 0.5, which represents the limit where the channels are unconditionally stable (real roots).

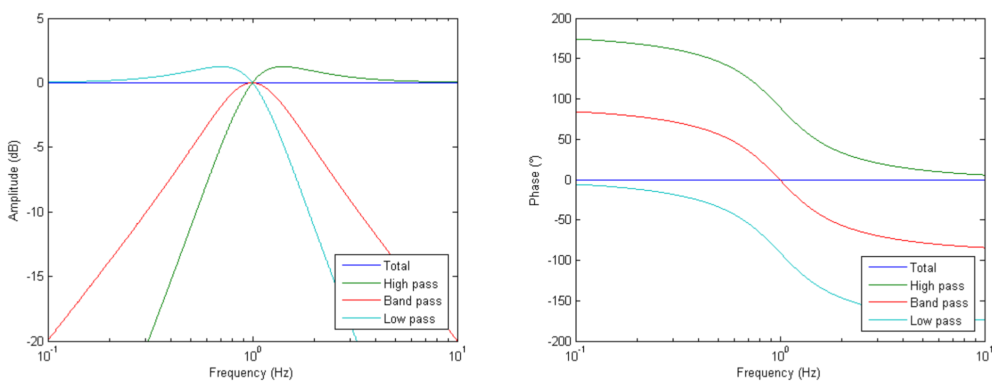


Figure 17 – Amplitude and phase response with a_1 = 1.

The phase difference between the neighbour channels is 90° throughout frequency and 180° between the low-pass and high-pass channels.

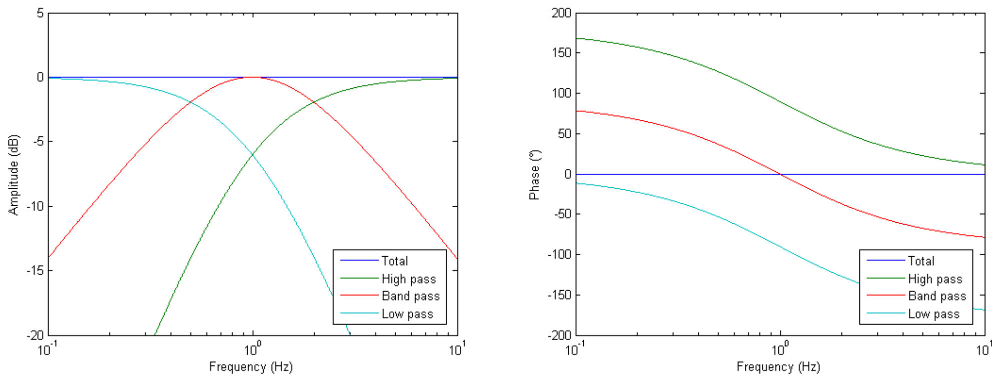


Figure 18 – Amplitude and phase response with $a_1 = 2$.

2.2.4. Steep cut-off – two way

Maximum cut-off slope of both channels is obtained if the first-order term is removed from the nominator, resulting in a two-way crossover. The filter does not satisfy the requirement of unity transfer function since $H_2 \neq 1$ but it can realise an all-pass filter.

$$\boxed{\begin{aligned} H_{HP2} &= \frac{s_0^2}{1 + a_1 s_0 + s_0^2} \\ H_{LP2} &= \frac{1}{1 + a_1 s_0 + s_0^2} \end{aligned}}$$

Determination of the coefficients assume modelling by a combination of two first-order Butterworth filters in cascade (the Linkwitz-Riley method). The nominator polynomial is not important for this evaluation, only the denominator polynomial is considered.

$$\begin{aligned} H_{LP2} = H_{LP1}^2 &= \left(\frac{N_1}{1 + s_0} \right)^2 \\ &= \frac{N_1^2}{1 + 2s_0 + s_0^2} \end{aligned}$$

By comparison, the coefficient is found to:

$$a_1 = 2$$

The transfer functions of the channels become:

$$H_{LP2} = \frac{1}{1 + 2s_0 + s_o^2} \quad and \quad H_{HP2} = \frac{s_0^2}{1 + 2s_0 + s_o^2}$$

The resultant transfer function becomes:

$$H_2 = \frac{1 + s_0^2}{1 + 2s_0 + s_o^2}$$

At crossover is $s_0 = \mathrm{i}$ so the nominator equates zero; i.e. the filter introduces a notch at the crossover frequency as can be seen from Figure 20.

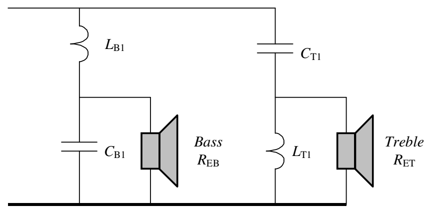


Figure 19 – Passive crossover network.

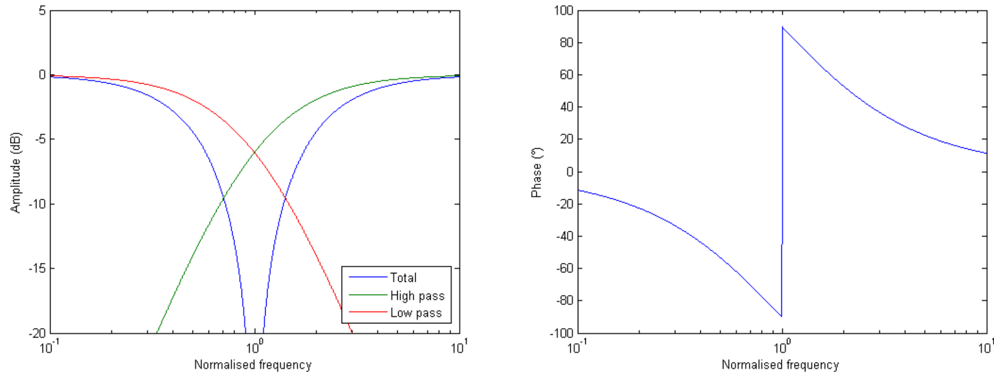


Figure 20 – Amplitude phase response with a_1 = 2. The notch at crossover is due to the phase difference between the loudspeaker. The bass loudspeaker is –90° and the treble loudspeaker is 90°, i.e. 180° apart, so the outputs cancel.

A solution is to invert the polarity of one of the channels, often the treble loudspeaker, which restores the phase difference to 0°. The resultant transfer function becomes:

$$H_2 = H_{LP2} - H_{HP2} = \frac{1 - s_0^2}{1 + a_1 s_0 + s_0^2}$$

The resulting amplitude response is flat but the phase decreases gradually from 0° at low frequencies to –180° at high frequencies. This is a small price to pay for a filter with sufficient cut-off slope to reduce the bandwidth requirement to 1 octave outside cut-off and to protect the treble loudspeaker against low-frequency signals.

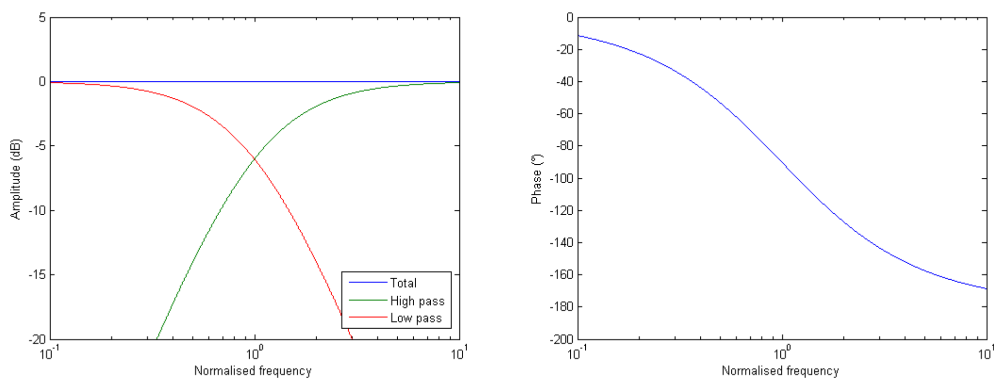


Figure 21 – Amplitude and phase response with a_1 = 2 and inverted treble channel.

The introduction of a phase different from zero introduces a group delay, which is shown in Figure 22. Note that the group delay unit is calculated for a normalised filter corresponding to a cut-off frequency of 1 Hz. With a cut-off frequency of 1000 Hz the group delay scaling will be in milliseconds and not seconds.

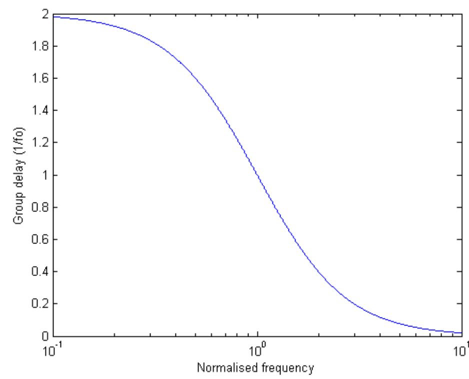


Figure 22 – Group delay with a_1 = 2 and inverted treble channel.

The filter is very popular due to the low component count of two components per channel and relative steep cut-off.

2.3. Third order

Ideal crossover networks of third order are problematic, as will be shown in this section. The problem being large phase difference between channels, which results in quite odd designs.

A third-order transfer function is defined from H_N with $N = 3$:

$$H_3 = \frac{1 + a_1 s_0 + a_2 s_0^2 + s_0^3}{1 + a_1 s_0 + a_2 s_0^2 + s_0^3}$$

Determination of the coefficients assume modelling by a combination of a first-order filter and a second-order filter in cascade. The nominator polynomial is not important for this evaluation, only the denominator polynomial is considered.

$$\begin{aligned} H_{LP3} = H_{LP1} \times H_{LP2} &= \frac{N_1}{1+s} \times \frac{N_2}{1 + cs + s_0^2} \\ &= \frac{N_1 N_2}{1 + (1+c)s_0 + (1+c)s_0^2 + s_0^3} \end{aligned}$$

By comparison, the coefficients are found to:

$$\begin{aligned} a_1 &= 1 + c \\ a_2 &= 1 + c \end{aligned}$$

The value of c must be chosen for the use with all-pass filters. The phase difference between the channels is 270° at crossover (since $i^3 = -1$), so the signals are added with a loss of 3 dB. The crossover filters should thus be –3 dB and since the first-order filter realises this, the second-order filter must be set to 0 dB at crossover so it requires $c = 1$ since $c = 1/Q$ for the second-order filter defines the level at resonance (equal to Q).

2.3.1. Asymmetrical – two way

One obvious realisation of a two-way crossover network is to divide between the first-order and second-order terms:

$$\boxed{\begin{aligned} H_{HP3} &= \frac{s_0^3}{1 + a_1 s_0 + a_2 s_0^2 + s_0^3} \\ H_{LP3} &= \frac{1 + a_1 s_0 + a_2 s_0^2}{1 + a_1 s_0 + a_2 s_0^2 + s_0^3} \end{aligned}}$$

A passive implementation is sensitive to loudspeaker impedance because of the series elements. The damping must be supplied by the load resistance, the loudspeaker, which is far from resistive if not compensated properly, so the passive crossover network requires impedance compensation of both branches.

At crossover, and assuming $a_1 = a_2 = 2$ for simplicity, we get:

$$H_{HP3} = \frac{-i}{1+2i-2-i} = \frac{-i}{-1+i} = \frac{i}{1-i} = 0.71\angle+135°$$
$$H_{LP3} = \frac{1+2i-2}{1+2i-2-i} = \frac{-1+2i}{-1+i} = \frac{1-2i}{1-i} = 1.58\angle-18°$$

So, the channels are 153° out of phase at crossover but the level is increased for the low-frequency channels to compensate for the loss and the channels add up to 0°. This is not a healthy way of designing a crossover network, the design should not be based upon subtraction of large figures; this will easily lead to problems.

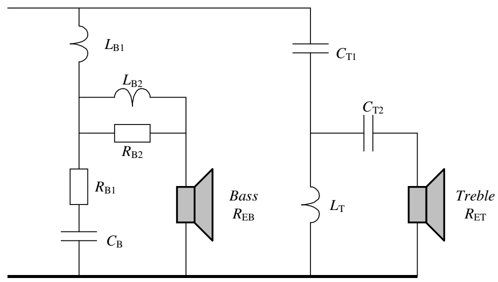


Figure 23 – Passive crossover network.

An active filter realisation could use the following algorithm to extract the low-pass channel from the high-pass channel.

$$H_{LP3} = 1 - H_{HP3}$$

The resulting amplitude response is shown below for two arbitrarily selected values of the coefficients a_1 and a_2.

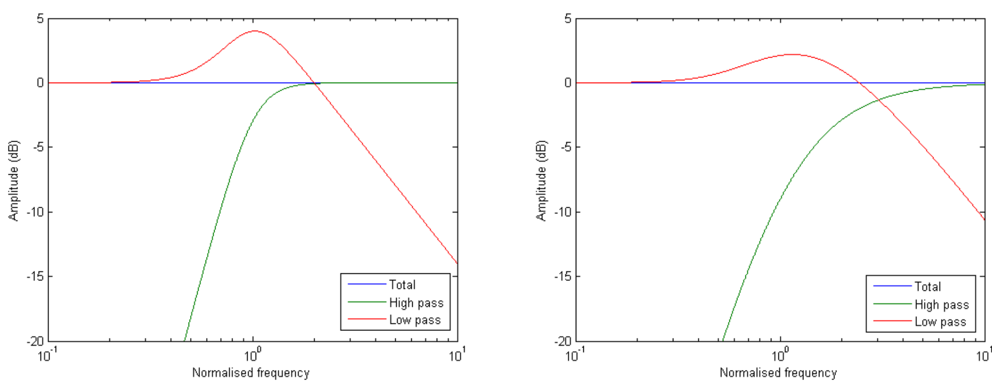


Figure 24 – Amplitude response with $a_1 = a_2 = 2$ (left) and $a_1 = a_2 = 3$ (right).

The peaking becomes worse for smaller values and the slope becomes too soft for larger values of a_1 and a_2 so this is not a particularly valuable design. The high-pass filter cut-off slope is 18 dB/octave, but the low-pass channel is only –6 dB/octave so the loudspeaker must be well-behaved three octaves above the cut-off frequency.

2.3.2. Symmetrical – two way

One obvious realisation of a two-way crossover network is to divide between the second-order and third-order terms:

$$\boxed{\begin{aligned} H_{HP3} &= \frac{a_2 s_0^2 + s_0^3}{1 + a_1 s_0 + a_2 s_0^2 + s_0^3} \\ H_{LP3} &= \frac{1 + a_1 s_0}{1 + a_1 s_0 + a_2 s_0^2 + s_0^3} \end{aligned}}$$

Cut-off slope is ±12 dB for both channels and the filter is symmetrical for $a_1 = a_2$. At crossover, and assuming $a_1 = a_2 = 2$ for simplicity, we get:

$$H_{HP3} = \frac{-2-i}{1+2i-2-i} = \frac{-2-i}{-1+i} = \frac{2+i}{1-i} = 1.58\angle+72°$$

$$H_{LP3} = \frac{1+2i}{1+2i-2-i} = \frac{1+2i}{-1+i} = -\frac{1+2i}{1-i} = 1.58\angle-72°$$

So, the channels are 144° out of phase at crossover but the level is increased for both channels to compensate for the loss and the channels add up to 0°. This is not a healthy way of designing a crossover network, the design should not be based upon subtraction of large figures; this will easily lead to problems.

A passive implementation is sensitive to loudspeaker impedance because of the series elements. The damping must be supplied by the load resistance, the loudspeaker, which is far from resistive if not compensated properly.

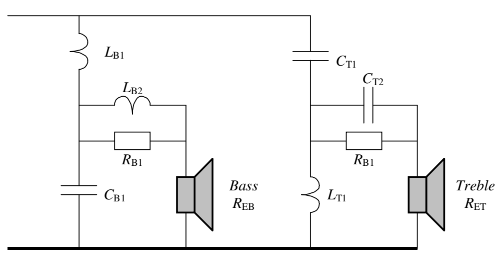


Figure 25 – Passive crossover network.

An active filter realisation could use the following algorithm to extract the low-pass channel from the high-pass channel.

$$H_{LP3} = 1 - H_{HP3}$$

The resulting amplitude response is shown below for two arbitrarily selected values of the coefficients a_1 and a_2. The cut-off slope is sufficient to reduce the loudspeaker requirement to 2 octaves past the crossover frequency and using a value of $a_1 = a_2$ around 4 seems useful since the peaking is limited to around 1 dB for each channel, which could be expected to work in real life.

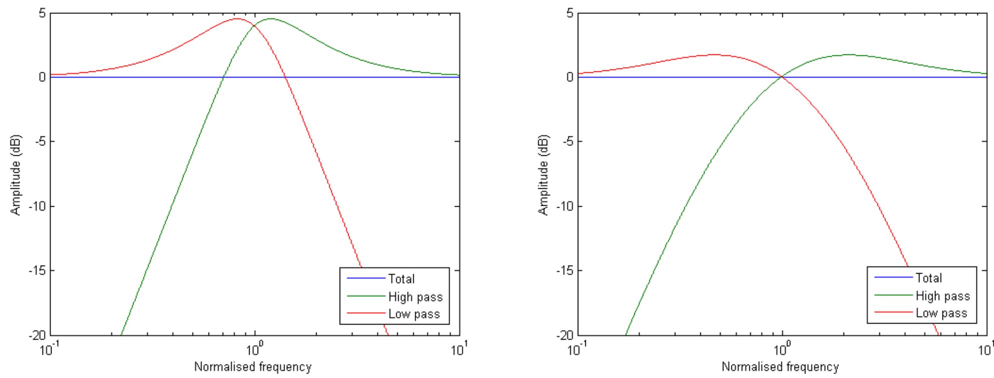


Figure 26 – Amplitude response with $a_1 = a_2 = 2$ (left) and $a_1 = a_2 = 3.7$ (right).

2.3.3. Symmetrical – three way

A symmetrical three-way crossover network can be build by using the middle two terms for the midrange loudspeaker:

$$\boxed{\begin{aligned} H_{HP3} &= \frac{s_0^3}{1 + a_1 s_0 + a_2 s_0^2 + s_0^3} \\ H_{BP3} &= \frac{a_1 s_0 + a_2 s_0^2}{1 + a_1 s_0 + a_2 s_0^2 + s_0^3} \\ H_{LP3} &= \frac{1}{1 + a_1 s_0 + a_2 s_0^2 + s_0^3} \end{aligned}}$$

Cut-off slope is ±18 dB/octave for the bass and treble channels but only ±6 dB/octave for the midrange channel. The filter will be symmetrical for $a_1 = a_2$.

A passive implementation is not attractive but the active solution is straightforward, when the low-pass and high-pass channels have been constructed:

$$H_{BP3} = 1 - H_{LP3} - H_{HP3}$$

The low-pass and high-pass channels can be constructed from standard third-order Butterworth filter blocks – but the design is quite tricky as the following analysis will show. Assume that the coefficients are $a_1 = a_2 = 2$, to simplify the analysis. The following amplitudes and phases can then be found at the crossover frequency ($s_0 = \mathrm{i}$):

$$\begin{aligned} H_{HP3} &= \frac{-i}{1 + 2i - 2 - i} = \frac{-i}{-1 + i} = \frac{i}{1 - i} = 0.71\angle + 135^\circ \\ H_{BP3} &= \frac{2i - 2}{1 + 2i - 2 - i} = 2\frac{i - 1}{-1 + i} = 2\frac{1 - i}{1 - i} = 2\angle 0^\circ \\ H_{LP3} &= \frac{1}{1 + 2i - 2 - i} = \frac{1}{-1 + i} = \frac{-1}{1 - i} = 0.71\angle - 135^\circ \end{aligned}$$

So, the low-pass and high-pass channels combine to –1 at crossover while the midrange channel is at 0°, which means that the two channels opposes the midrange channel. The midrange channel must be boosted to 2 in order for the sum of all channels to be unity.

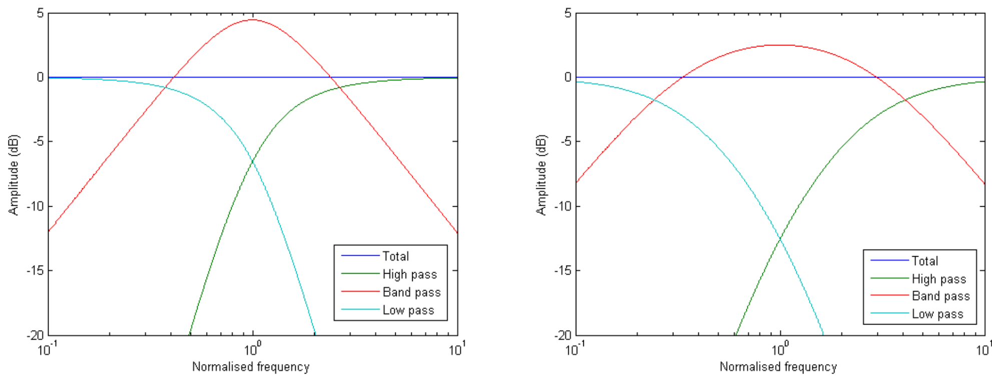


Figure 27 - Amplitude response for the symmetrical third-order three-way crossover network with $a_1 = a_2 = 2.5$ (left) and $a_1 = a_2 = 4$ (right).

The conclusion is, that the design would do better without the midrange channel, and this is exactly the following crossover network to be analysed. This is at the end of the ideal filters with H = 1, since higher order filters includes too many terms; they are cumbersome to implement, especially with passive filters.

2.3.4. Steep cut-off – two way

In order to improve the cut-off slope of the low-pass channel one method is to remove the a_1 and a_2 coefficients of the nominator:

$$\boxed{\begin{aligned} H_{HP3} &= \frac{s_0^3}{1 + a_1 s_0 + a_2 s_0^2 + s_0^3} \\ H_{LP3} &= \frac{1}{1 + a_1 s_0 + a_2 s_0^2 + s_0^3} \end{aligned}}$$

An active filter realisation could use the following algorithm to extract the low-pass channel from the high-pass channel.

$$H_{LP3} = 1 - H_{HP3}$$

Again, the passive filter must use impedance compensation of the loudspeakers.

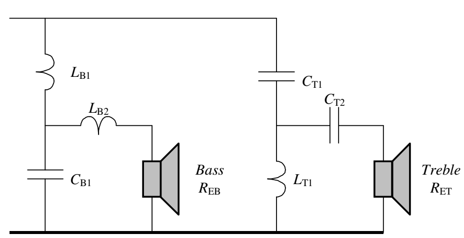


Figure 28 – Passive crossover network.

The coefficients must be $a_1 = a_2 = 2$ for the all-pass filter and the result is shown in Figure 29. The phase does not jump 360° at crossover; this is due to the MATLAB *angle* function.

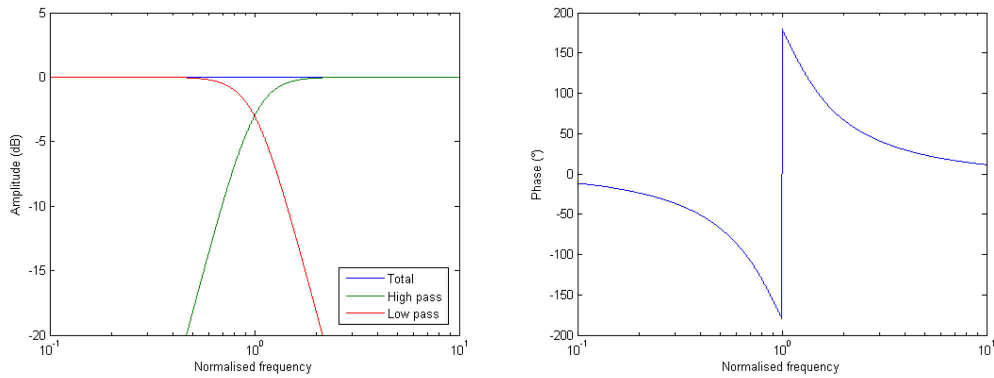


Figure 29 - Amplitude and phase response with $a_1 = a_2 = 2$.

The introduction of a phase different from zero introduces a group delay shown in Figure 30. Note that the group delay unit is calculated for a normalised filter corresponding to a cut-off frequency of 1 Hz. With a cut-off frequency of 1000 Hz the group delay scaling will be in milliseconds and not seconds.

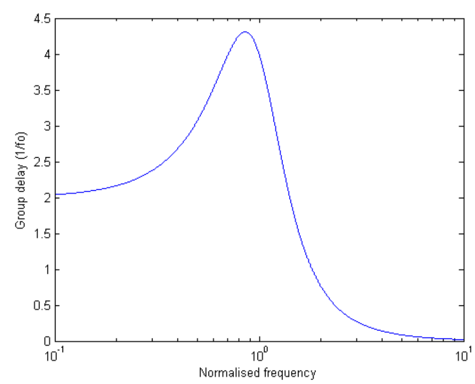


Figure 30 – Group delay with $a_1 = a_2 = 2$.

The filter includes two coefficients and its sensitivity to variations was analysed by scaling the coefficients by ±10 % with the result shown in Figure 31.

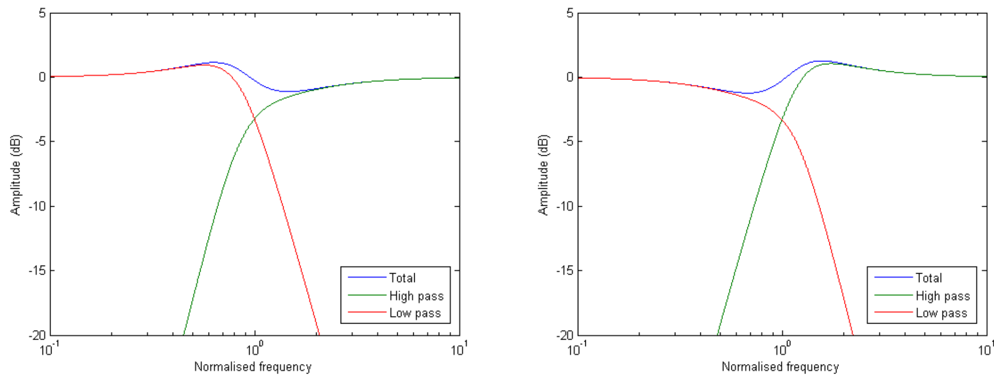


Figure 31 - Amplitude response for 10 % change of coefficients: a_1 = 1.8, a_2 = 2.2 (left) and a_1 = 2.2, a_2 = 1.8 (right).

The filter accept inversion of the treble loudspeaker.

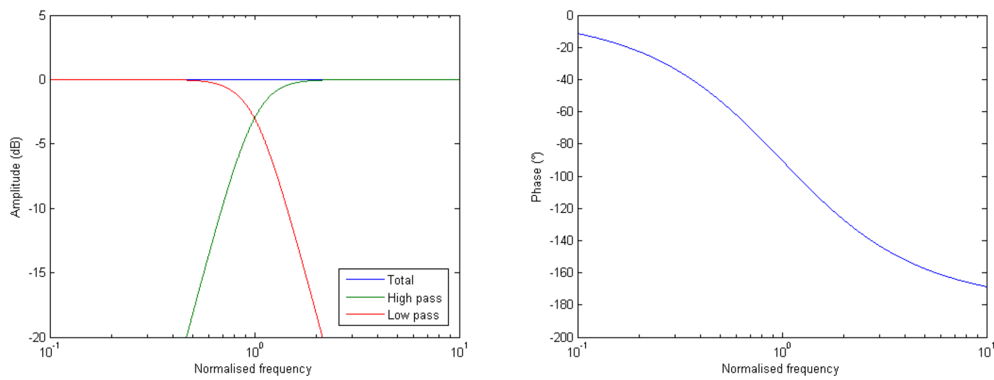


Figure 32 - Amplitude and phase response with $a_1 = a_2 = 2$ and inverted treble.

The group delay is reduced in amplitude and becomes monotonically as result of the inversion, so although one may argument against the inversion, there could be an audible improvement by doing so.

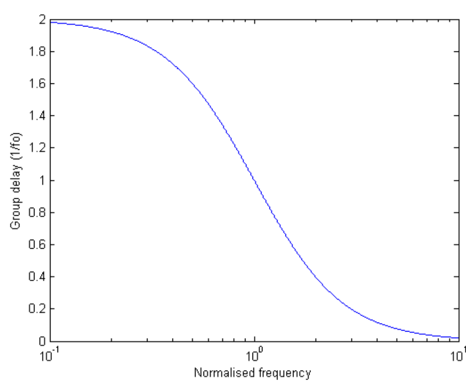


Figure 33 – Group delay with $a_1 = a_2 = 2$ and inverted treble loudspeaker.

2.4. Fourth order

Crossover networks of fourth-order are popular due to the steep cut-off slopes, which reduces the loudspeaker requirement to less than 2 octaves beyond the crossover frequency. However, a passive implementation is practical only for the all-pass filter, since component count would be too large with the three-way system.

A fourth-order transfer function is defined from H_N with $N = 4$:

$$H_4 = \frac{1 + a_1 s_0 + a_2 s_0^2 + a_3 s_0^3 + s_0^4}{1 + a_1 s_0 + a_2 s_0^2 + a_3 s_0^3 + s_0^4}$$

2.4.1. Symmetrical – three way

A symmetrical three-way crossover network can be build using the middle term for the midrange loudspeaker. Coefficient a_2 must be removed from the nominator.

$$\boxed{\begin{aligned} H_{HP4} &= \frac{a_3 s_0^3 + s_0^4}{1 + a_1 s_0 + a_2 s_0^2 + a_3 s_0^3 + s_0^4} \\ H_{BP4} &= \frac{a_2 s_0^2}{1 + a_1 s_0 + a_2 s_0^2 + a_3 s_0^3 + s_0^4} \\ H_{LP4} &= \frac{1 + a_1 s_0}{1 + a_1 s_0 + a_2 s_0^2 + a_3 s_0^3 + s_0^4} \end{aligned}}$$

An active implementation could use:

$$H_{BP4} = 1 - H_{LP4} - H_{HP4}$$

The low-pass and high-pass channels includes two terms each so they cannot be implemented using standard low-pass and high-pass filters but the circuitry is not too complex to be implemented using active filters. When build, the midrange channel is derived by subtracting the channels from the input signal.

Assume that the coefficients are $a_1 = 2$, $a_2 = 3$ and $a_3 = 2$, to simplify the analysis. The following amplitudes and phases can then be found at the crossover frequency (s_0 = i):

$$H_{HP3} = \frac{-2i + 1}{1 + 2i - 3 - 2i + 1} = \frac{1 - 2i}{-1} = 2.2\angle 243°$$
$$H_{BP3} = \frac{-3}{1 + 2i - 3 - 2i + 1} = \frac{-3}{-1} = 3.0\angle 0°$$
$$H_{LP3} = \frac{1 + 2i}{1 + 2i - 3 - 2i + 1} = \frac{1 + 2i}{-1} = 2.2\angle -243°$$

So, the low-pass and high-pass channels are 486° out of phase (corresponds to 126°) and adds with some loss and the midrange channel adds the required signal. The channels are all at fairly high levels around crossover so this is a design, which is based upon subtraction of large figures – it should be avoided.

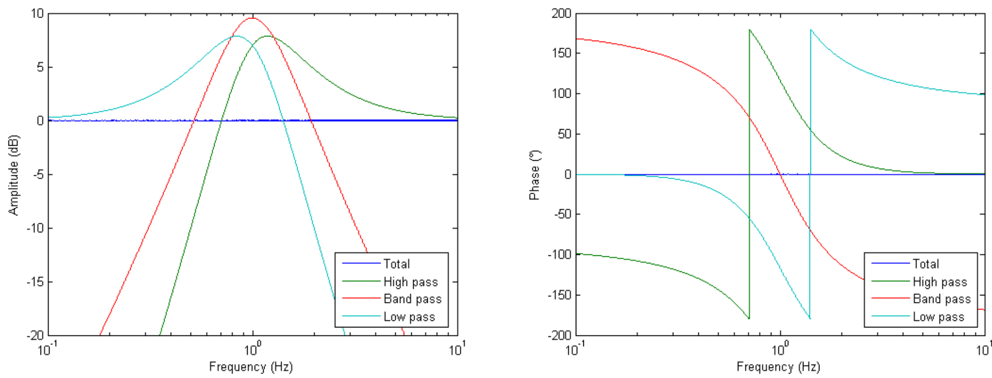


Figure 34 – Amplitude and phase with $a_1 = 2$, $a_2 = 3$, $a_3 = 2$.

2.4.2. Steep cut-off – two way

The terms with a_1, a_2 and a_3 are removed from the nominator resulting in a two-way crossover with maximum cut-off slope within both channels. The filter does not satisfy the requirement of unity transfer function.

$$\boxed{\begin{aligned} H_{HP4} &= \frac{s_0^4}{1 + a_1 s_0 + a_2 s_0^2 + a_3 s_0^3 + s_0^4} \\ H_{LP4} &= \frac{1}{1 + a_1 s_0 + a_2 s_0^2 + a_3 s_0^3 + s_0^4} \end{aligned}}$$

Determination of the coefficients assume modelling by a combination of two second-order Butterworth filters in cascade (the Linkwitz-Riley method). The nominator polynomial is not important for this evaluation, only the denominator is considered.

$$\begin{aligned} H_{LP4} = H_{LP2} \times H_{LP2} &= \frac{N_1}{1 + cs_0 + s_0^2} \times \frac{N_2}{1 + cs_0 + s_0^2}, \quad where\ c = \sqrt{2} \\ &= \frac{N_1 N_2}{1 + 2cs_0 + \left(2 + c^2\right)s_0^2 + 2cs_0^3 + s_0^4} \end{aligned}$$

By comparison, the coefficients are found to:

$$\begin{aligned} a_1 &= 2c &&= 2.83 \\ a_2 &= 2 + c^2 &&= 4.00 \\ a_3 &= 2c &&= 2.83 \end{aligned}$$

A passive implementation is at the limit of what can (or should) be done but the network is straight forward from a theoretical point of view. It is a requirement that the loudspeakers are impedance compensated.

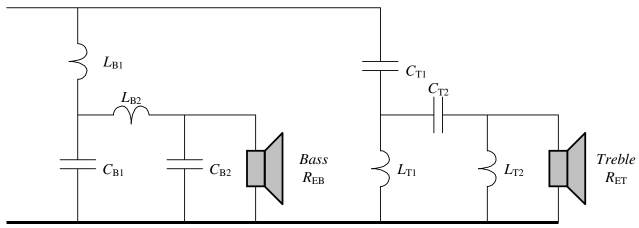


Figure 35 – Passive crossover network.

The result is shown in Figure 36. The phase difference between the channels are 360° at crossover so the signals are added without loss.

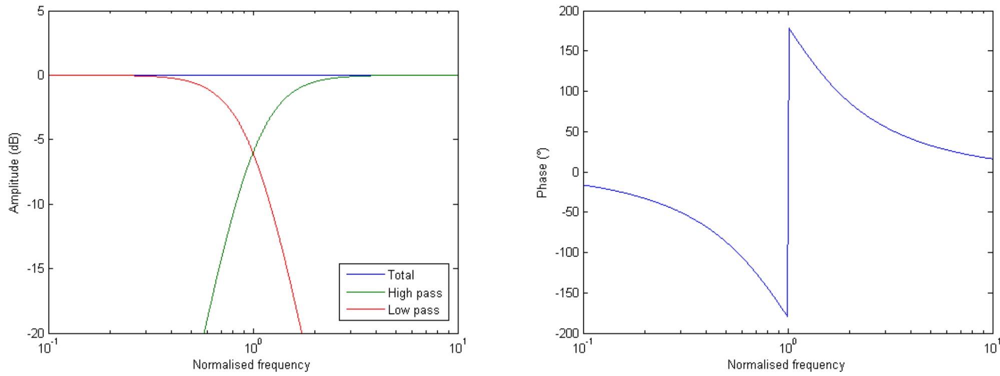


Figure 36 – Amplitude and phase response with $a_1 = a_3 = 2.83$, $a_2 = 4.00$.

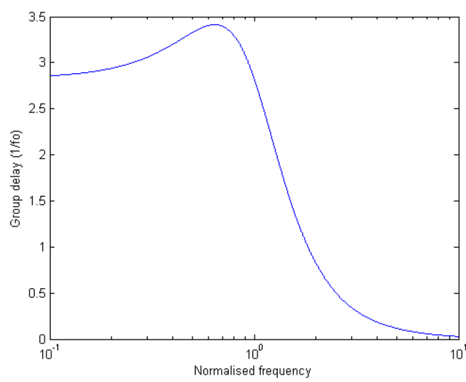


Figure 37 – Group delay with $a_1 = a_3 = 2.83$, $a_2 = 4.00$.

2.5. Passive network

A passive network is best suited for low order crossover networks and can be build as shown in Figure 38, where Z_1, Z_2, etc. are impedances, which may consist of resistors, inductors and capacitors or even combinations hereof. The number of branches is defined by the filter order; a first order crossover network would consist of Z_1 only.

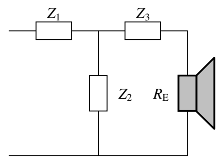


Figure 38 – Conventional ladder-network for a passive crossover. The impedances can be any of resistor, inductor or capacitor.

The transfer function of the filter can be derived from inspection of the circuitry. The below collection of crossover network transfer functions is limited to third order since higher order networks become more involved and are of little practical use. Higher order filters should preferably be build using active circuitry.

2.5.1. First order

This consists only of Z_1 so Z_2 not used and Z_3 is a short circuit. The filter is a voltage divider between the series element Z_1 and the loudspeaker, represented by the DC voice coil resistance R_E. The transfer function for the first order network is:

$$H_1 = \frac{R_E}{Z_1 + R_E}$$

To introduce real components, the impedance must be substituted by an inductor or a capacitor. The impedance of the inductor and capacitor is defined as:

$$Z_L = sL$$
$$Z_C = \frac{1}{sC}$$

Using an inductor for Z_1 the result is a low-pass filter:

$$H_{1LP} = \frac{R_E}{sL + R_E} = \frac{1}{1 + \frac{sL}{R_E}} = \frac{1}{1 + s_0}, \quad where\ \omega_0 = \frac{R_E}{L}$$

Using a capacitor for Z_1 the result is a high-pass filter:

$$H_{1HP} = \frac{R_E}{\frac{1}{sC} + R_E} = \frac{sCR_E}{1 + sCR_E} = \frac{s_0}{1 + s_0}, \quad where\ \omega_0 = \frac{1}{CR_E}$$

2.5.2. Second order

The crossover network includes Z_2, which is shunted across the loudspeaker (since Z_3 is a short circuit), so the transfer function can be derived from H_1 by substituting R_E by a parallel combination of Z_2 and R_E:

$$H_2 = \frac{\dfrac{Z_2 R_E}{Z_2 + R_E}}{Z_1 + \dfrac{Z_2 R_E}{Z_2 + R_E}} = \frac{Z_2}{\dfrac{Z_1 Z_2}{R_E} + Z_1 + Z_2}$$

Using an inductor for Z_1 and a capacitor for Z_2 the result is a low-pass filter:

$$H_{2LP} = \frac{\dfrac{1}{sC}}{sL\dfrac{1}{sCR_E} + sL + \dfrac{1}{sC}}$$

After reduction:

$$H_{2LP} = \frac{1}{1 + a_1 s_0 + s_0^2}, \quad where\ \omega_0 = \frac{1}{\sqrt{CL}} \quad a_1 = \frac{1}{Q} = \frac{1}{R_E}\sqrt{\frac{L}{C}}$$

Using a capacitor for Z_1 and an inductor for Z_2 the result is a high-pass filter:

$$H_{2HP} = \frac{sLR_E}{\dfrac{1}{sC}sL + \dfrac{1}{sC}R_E + sLR_E}$$

After reduction:

$$H_{2HP} = \frac{s_0^2}{1 + a_1 s_0 + s_0^2}, \quad where\ \omega_0 = \frac{1}{\sqrt{CL}} \quad a_1 = \frac{1}{Q} = \frac{1}{R_E}\sqrt{\frac{L}{C}}$$

2.5.3. Third order

Derivation starts from the loudspeaker where Z_3 forms a voltage divider with R_E. The two first components also forms a voltage divider between Z_1 and Z_2, where Z_2 is in parallel with $Z_3 + R_E$.

$$H_3 = \frac{\dfrac{Z_2(Z_3 + R_E)}{Z_2 + Z_3 + R_E}}{Z_1 + \dfrac{Z_2(Z_3 + R_E)}{Z_2 + Z_3 + R_E}} \frac{R_E}{Z_3 + R_E}$$

After reduction::

$$H_3 = \frac{Z_2 R_E}{Z_1 Z_2 + Z_1 Z_3 + Z_1 R_E + Z_2 Z_3 + Z_2 R_E}$$

Using an inductor for Z_1 and Z_3 and a capacitor for Z_2 the result is a low-pass filter:

$$H_{3LP} = \frac{\frac{1}{sC}R_E}{sL\frac{1}{sC} + (sL)^2 + sLR_E + \frac{1}{sC}sL + \frac{1}{sC}R_E} = \frac{1}{s\frac{L}{R_E} + s^3\frac{CL^2}{R_E} + s^2CL + \frac{sL}{R_E} + 1}$$

After reduction:

$$H_{3LP} = \frac{1}{1 + a_1 s_0 + a_2 s_0^2 + s_0^3}$$

$$\omega_0 = \sqrt[3]{\frac{R_E}{CL^2}} \quad a_1 = \frac{1}{R_E}\sqrt[3]{\frac{LR_E}{C}} \quad a_2 = CL\left(\sqrt[3]{\frac{R_E}{CL^2}}\right)^2$$

Using a capacitor for Z_1 and Z_3 and an inductor for Z_2 the result is a high-pass filter:

$$H_3 = \frac{sLR_E}{\frac{1}{sC}sL + \left(\frac{1}{sC}\right)^2 + \frac{1}{sC}R_E + sL\frac{1}{sC} + sLR_E} = \frac{s^2CL}{s\frac{L}{R_E} + \frac{1}{sCR_E} + 1 + s\frac{L}{R_E} + s^2CL}$$

After reduction:

$$H_{3LP} = \frac{s_0^3}{1 + a_1 s_0 + a_2 s_0^2 + s_0^3}$$

$$\omega_0 = \sqrt[3]{\frac{R_E}{CL^2}} \quad a_1 = \frac{1}{R_E}\sqrt[3]{\frac{LR_E}{C}} \quad a_2 = CL\left(\sqrt[3]{\frac{R_E}{CL^2}}\right)^2$$

2.5.4. Fourth order

Filter realisation will use the ladder-network shown in Figure 39, which is a general filter using impedances.

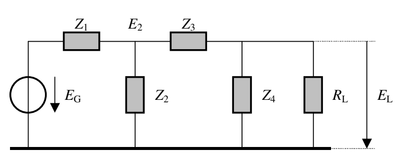


Figure 39 – Generic ladder-filter for realisation of the crossover networks. The impedance *Z* represents any combination of capacitors (*Z* = 1/*sC*), inductors (*Z* = *sL*) or resistors (*Z* = *R*). The load resistor R_L represents the loudspeaker.

The voltages at the nodes are, according to Kirchhoff's law, which states that the sum of currents to a node must equal zero (with positive direction defined away from the node):

$$\frac{E_2 - E_G}{Z_1} + \frac{E_2}{Z_2} + \frac{E_2 - E_L}{Z_3} = 0$$

$$\frac{E_L - E_2}{Z_3} + \frac{E_L}{Z_4} + \frac{E_L}{R_L} = 0$$

The terms are rearranged:

$$\left(\frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}\right)E_2-\frac{E_G}{Z_1}-\frac{E_L}{Z_3}=0$$

$$\left(\frac{1}{Z_3}+\frac{1}{Z_4}+\frac{1}{R_L}\right)E_L-\frac{E_2}{Z_3}=0 \quad \Rightarrow \quad E_2=\left(1+\frac{Z_3}{Z_4}+\frac{Z_3}{R_L}\right)E_L$$

Elimination of E_2 results in:

$$\left(\frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}\right)\left(1+\frac{Z_3}{Z_4}+\frac{Z_3}{R_L}\right)E_L-\frac{E_L}{Z_3}=\frac{E_G}{Z_1}$$

And:

$$\left[\left(\frac{1}{Z_1}+\frac{1}{Z_2}+\frac{1}{Z_3}\right)\left(1+\frac{Z_3}{Z_4}+\frac{Z_3}{R_L}\right)-\frac{1}{Z_3}\right]E_L=\frac{E_G}{Z_1}$$

And:

$$\left[\left(1+\frac{Z_1}{Z_2}+\frac{Z_1}{Z_3}\right)\left(1+\frac{Z_3}{Z_4}+\frac{Z_3}{R_L}\right)-\frac{Z_1}{Z_3}\right]E_L=E_G$$

The transfer function becomes:

$$\boxed{H=\frac{E_L}{E_G}=\frac{1}{\left(1+\frac{Z_1}{Z_2}+\frac{Z_1}{Z_3}\right)\left(1+\frac{Z_3}{Z_4}+\frac{Z_3}{R_L}\right)-\frac{Z_1}{Z_3}}}$$

2.5.5. Loudspeaker impedance

All passive crossover networks are sensitive to the load impedance and they are most often designed for constant and resistive loading; but the impedance of a loudspeaker is neither constant nor resistive as can be seen from Figure 40. The impedance is real only at very low frequencies and changes from capacitive to inductive within the pass band.

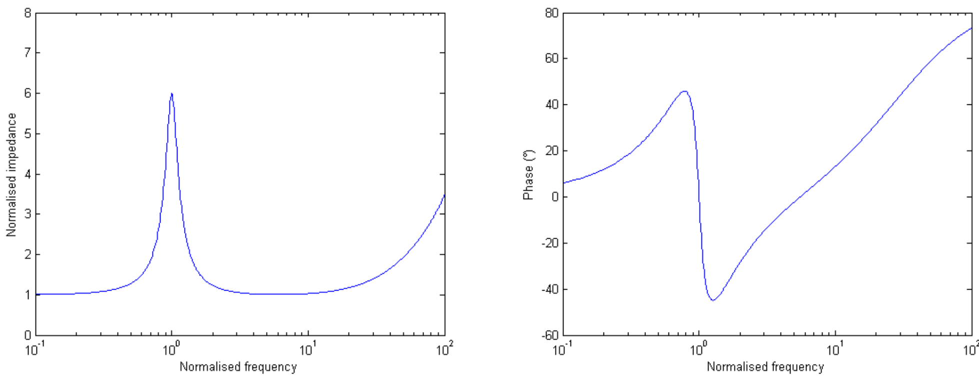


Figure 40 – Electrical impedance of loudspeaker with resonance frequency normalised to unity, with voice coil cut-off frequency set to 30 times the resonance frequency and with mechanical quality factor $Q_M = 5$.

A passive crossover network will not operate as intended if loaded by this impedance so the presumptions of the filter will be validated and the result can be rather dramatic.

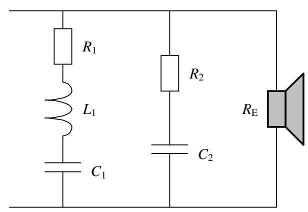


Figure 41 – Compensation networks for loudspeaker impedance correction.

Consider a treble loudspeaker with 1 kHz resonance frequency, which is to be activated above 4 kHz by a series capacitor. The required attenuation at the crossover frequency is (1 kHz)/(4 kHz) = 0.25, so the capacitor must have an impedance of 4 times the nominal impedance at this frequency. But the impedance of the loudspeaker may have increased to 4 times the nominal value, thus compensating for the rise in impedance; the effective crossover frequency becomes the resonance frequency of the loudspeaker.

It is possible to compensate, at least partially, for the variation in loudspeaker impedance, using networks shown Figure 41, thus introducing additional components increasing complexity and cost.

The equations for the correction networks are shown below [2]. The required parameters are the electrical and mechanical quality factors (Q_{EC} and Q_{MC}) and the resonance frequency (f_C) of the loudspeaker in the closed cabinet, the DC resistance of the voice coil (R_E), and the inductance of the voice coil (L_E).

$$R_1 = \left(1 + \frac{Q_{EC}}{Q_{MC}}\right) R_E$$

$$L_1 = \frac{R_E Q_{EC}}{2\pi f_C}$$

$$C_1 = \frac{1}{2\pi f_C R_E Q_{EC}}$$

$$R_1 = R_E$$

$$C_2 = \frac{L_E}{2\pi R_E^2}$$

Midrange loudspeaker: Q_{EC} = 0.4, Q_{MC} = 3, f_S = 80 Hz, R_E = 6 Ω.and L_E = 0.1 mH. Compensation of the resonance frequency: R_1 = 6.8 Ω, L_1 = 4.8 mH and C_1 = 800 µF. Compensation of the voice coil inductance: 6 Ω and 0.44 µF.

It is not required compensating for the resonance frequency of a bass loudspeaker, since this is within the pass band of the crossover network and similar arguments are valid for the voice coil inductance of the treble loudspeaker. A midrange loudspeaker may require compensation of both parameters.

An active crossover network solves the problem of interaction between the crossover network and the loudspeaker. It offers better control of filter parameters, since inductors can be avoided, and it is insensitive to the temperature dependency of the voice coil DC impedance. Problems, which must be addressed by the designer of a passive crossover network, in addition to the more obvious problems of selecting the crossover frequencies, deciding a network topology and how to construct the loudspeaker system from available components.

2.6. Active network

Active networks are build from filter blocks, such as the two blocks shown in Figure 42. Amplifiers are used as buffers avoiding interaction between the filter sections. It should be mentioned that the layout possibilities are large and the below examples represents but a few of the possible implementations.

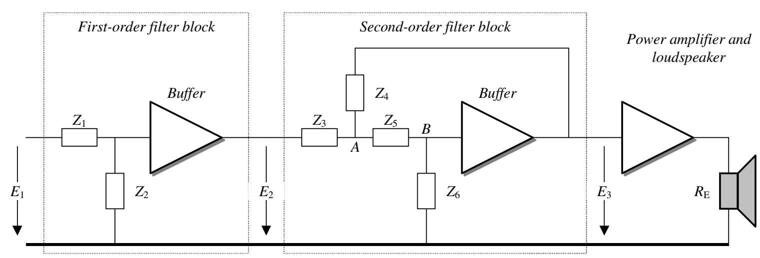


Figure 42 – Active filter consisting of a first-order filter block and a second-order filter block. The amplifiers isolates the sections thus simplifying the design.

Active filters are designed from resistors, capacitors and amplifier circuits. The inductor is not required and has been omitted since it is difficult to build good inductors; they are plagued by their series resistance and parallel capacitance and the component is prone to pick-up of hum from magnetic fields. The resistor and capacitor are almost ideal components with few parasitic components. The missing inertia, offered by the inductor in the passive crossover networks, and required by circuits with complex poles, is supplied by the amplifier.

In this example is the amplifier arranged as a voltage-follower, which means that it monitors the voltage at the input without loading the circuitry and outputs the voltage. The gain factor (amplification factor) is unity.

2.6.1. First order

The filter is actually a passive filter since one of the components are a resistor and the other a capacitor. The network is energised from node E_1 with output at node E_2.

$$H_1 = \frac{E_2}{E_1} = \frac{Z_2}{Z_1 + Z_2} \xrightarrow[Z_2 = R]{Z_1 = 1/sC} H_{1HP} = \frac{sCR}{1 + sCR} = \frac{s_0}{1 + s_0}$$

$$\xrightarrow[Z_2 = 1/sC]{Z_1 = R} H_{1LP} = \frac{1}{1 + sCR} = \frac{1}{1 + s_0}$$

$$\omega_0 = \frac{1}{CR}$$

2.6.2. Second order

The transfer function is best derived from the Kirchhoff law of nodes, which states that the sum of current into a node must equal zero. There are two internal nodes, A and B, but node B is identical to the output node E_3 due to the buffer. The network is energised from node E_2 with output at node E_3. The equations are:

$$\frac{E_A - E_2}{Z_3} + \frac{E_A - E_3}{Z_4} + \frac{E_A - E_3}{Z_5} = 0$$

$$\frac{E_3 - E_A}{Z_5} + \frac{E_3}{Z_4} = 0$$

After reduction:

$$H_2 = \frac{E_3}{E_2} = \frac{1}{1 + \frac{Z_3 + Z_5}{Z_6} + \frac{Z_3 Z_5}{Z_4 Z_6}}$$

Using resistors for Z_4 and Z_6 and capacitors for Z_3 and Z_5 results in a high-pass filter:

$$H_{2HP} = \frac{s_0^2}{1 + as_0 + s_0^2}, \quad where\ \omega_0 = \frac{1}{\sqrt{C_1 C_2 R_1 R_2}} \quad \frac{1}{a} = Q = \frac{\sqrt{C_1 C_2}}{C_1 + C_2}\sqrt{\frac{R_2}{R_1}}$$

Using resistors for Z_3 and Z_5 and capacitors for Z_4 and Z_6 results in a low-pass filter:

$$H_{2LP} = \frac{1}{1 + as_0 + s_0^2}, \quad where\ \omega_0 = \frac{1}{\sqrt{C_1 C_2 R_1 R_2}} \quad \frac{1}{a} = Q = \frac{\sqrt{R_1 R_2}}{R_1 + R_2}\sqrt{\frac{C_1}{C_2}}$$

2.6.3. Higher orders

Filters of higher order can be build from first and second order filter blocks by cascading the filters. A seventh order two-way crossover network will be used as an example:

$$H_{7HP} = \frac{s_0}{1 + s_0} \times \frac{s_0^2}{1 + a_1 s_0 + s_0^2} \times \frac{s_0^2}{1 + a_2 s_0 + s_0^2} \times \frac{s_0^2}{1 + a_3 s_0 + s_0^2}$$

$$H_{7LP} = \frac{1}{1 + s_0} \times \frac{1}{1 + a_1 s_0 + s_0^2} \times \frac{1}{1 + a_2 s_0 + s_0^2} \times \frac{1}{1 + a_3 s_0 + s_0^2}$$

All blocks share the same cut-off frequency, given by ω_0, with the Butterworth design but the coefficient a_1, a_2 and a_3, are different. The coefficients determine the roots of the denominator polynomial. The root of the first order polynomial is $s_0 = -1$, and the roots of the second order polynomials are determined from:

$$1 + as_0 + s_0^2 = 0$$

The roots are:

$$\alpha \pm i\beta = \frac{-a \pm \sqrt{a^2 - 4}}{2} = \frac{-a \pm i\sqrt{4 - a^2}}{2}$$

It follows that both the real part and imaginary parts of the roots (α and β respectively) are defined by the coefficient a. Hence, the value is given by:

$$a = -2\alpha$$

For the seventh order Butterworth polynomial are the roots defined as shown below in Table 2, which also shows the calculated values of the coefficients as well as the quality factors $Q = 1/a$:

Table 2 – Coefficients for a seventh order crossover network. The real value of the root determines the coefficient and thus the quality factor.

Section	Roots	Coefficient *a*	Quality factor *Q*
1	(-1)	-	-
2	(-0.2225 ±i0.9749)	0.445	2.247
3	(-0.6235 ±i0.7818)	1.247	0.802
4	(-0.9010 ±i0.4339)	1.802	0.555

The coefficient could as well be determined from the imaginary root value, which gives the same result. Take for instance the last root. With $a = 1.802$ is imaginary value:

$$\beta = \frac{\pm\sqrt{4 - a^2}}{2} = \frac{\pm\sqrt{4 - (1.802)^2}}{2} = \pm 0.4338$$

This is indeed the specified value.

2.6.4. Special

Several of the crossover networks used an algorithm such as the following for derivation of the low-pass channel from a steep high-pass channel.

$$H_{LP3} = 1 - H_{HP3}$$

This can relative easily be implemented as shown below where the input signal is high-pass filtered and routed through a power amplifier to the treble loudspeaker as well as to a subtraction circuitry for construction of the low-pass channel.

The high-pass filter is realised as a conventional third-order filter and a subtraction network (an operational amplifier) is used to generate the low-pass channel.

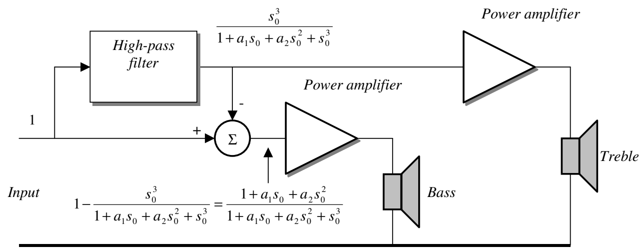


Figure 43 – Active crossover network.

3. Models

A loudspeaker is not just a linear transducer that outputs sound as a true replica of the input signal; the loudspeaker, the baffle and the listening room introduces limitations, severely affecting the signal quality. The loudspeaker bandwidth is limited and it is not small compared to wavelength at high frequencies, which affects the radiation angle concentrating the sound at the front of the loudspeaker. The front baffle introduces an impedance discontinuity at the edges, which cause reflections known as diffraction, that interferes with the direct signal and results in the well-known 6 dB loss of bass as well as ripples at higher frequencies. Listening off-axis introduces time difference for signals from the different loudspeakers, which create ripples around the crossover frequency, and reflections from large surfaces further affects the low frequency reproduction.

The model derived in this section is a *transfer function* for analytical studies using MATLAB. The models apply solely to the *electro-dynamic loudspeaker* using a moving-coil for the energy transfer; so electrostatic, piezoelectric and ribbon-type loudspeakers are not referenced.

At first, a model for the loudspeaker is introduced and this is followed by models for the surrounding; i.e. diffraction due to the baffle, the listening angle due to the distance between the loudspeakers and the reflections from the boundaries of the listening room. A model is also introduced for the calculation of group delay based upon the resultant transfer function.

3.1. Electro-acoustical model

The transfer function model for the loudspeaker consists of several terms each describing a specific part of the loudspeaker. The result is the sound pressure p at distance r with an number of parameters describing the loudspeaker.

$$\boxed{p = \left[H_{VC} \times H_D \times \frac{r_{REF}}{r} \exp(-ikr) \times K \right] \times E_G}$$

The excitation voltage E_G from the power amplifier is converted to a sound pressure by the constant K, which assembles the electrical, mechanical and acoustical parameters and represents the sound pressure within the middle of the pass-band at a reference distance r_{REF}, but it does not include parameters such as frequency and angle. The useful frequency range is specified by H_{VC} for the voice coil high-frequency cut-off and by H_D for the low frequency cut-off. The inverse-distance law is specified by r_{REF}/r, and the exponential function. The model can easily be enhanced by introducing more functions dealing with specific areas.

Derivation starts by analysing the mechanical construction of the loudspeaker and the following analysis will be based upon the introduction presented by *Leach* and with reference to *Beranek*. The loudspeaker model applies to bass, midrange and treble units and will be limited to units with the rear side radiating into a closed cavity thus realising a single model common to all the loudspeaker units.

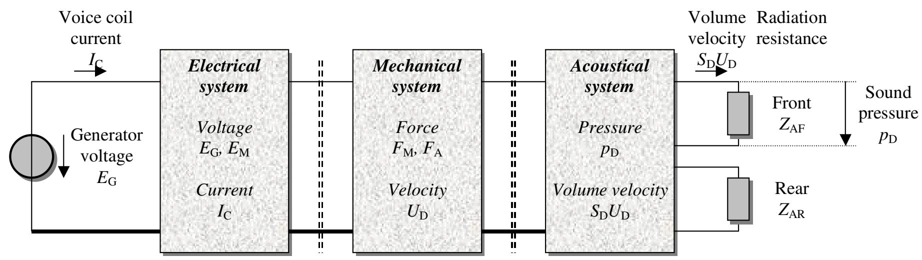


Figure 44 – The model is based upon the model of a loudspeaker and operates with three different system analogies: electrical, mechanical and acoustical.

The initial part of the analysis concentrates on deriving a model for the diaphragm velocity as a function of generator voltage, which introduces the loudspeaker frequency dependency. The diaphragm velocity is converted into sound pressure, which introduces the inverse-distance law. The remaining analysis deals with the consequence of diaphragm diameter, monitoring angle and the baffle size thus introducing directivity and diffraction. The consequence of reflections from the listening room will also be considered.

It is common to divide the loudspeaker unit into three different *system analogies* as shown in Figure 44. The electrical system is responsible for the voice coil current, the mechanical system translates this into diaphragm velocity and the acoustical system accounts for sound pressure, air loading and the effect of the closed cabinet.

3.1.1. The loudspeaker unit

A layout of the electro-dynamic loudspeaker is sketched in Figure 45, which also displays the main difference between the bass/midrange and treble units. The diaphragm of the treble unit is reduced to a dome to cut the mass and the size of the loudspeaker thus improving reproduction of the high-frequency range.

The diaphragm is the moving surface of the loudspeaker which radiates sound. The preferred material is paper, due to its low weight and high internal damping, although plastic and metal are also used. The diaphragm is cone or dome shaped to improve rigidity although loudspeakers with flat diaphragm exists. A suspension system is used to keep the diaphragm at the correct position with the voice coil within the magnet gap and restrict the diaphragm movement to the axial direction only.

The effective diameter of the diaphragm (D) includes some of the outer suspension, which is moving as well, so the effective cross-sectional area becomes:

$$S_D = \pi a^2 = \frac{\pi}{4} D^2$$

The area is 300 cm^2 for an 8 inch loudspeaker and 5 cm^2 for 1 inch diameter.

For treble loudspeakers is the diaphragm a dome and the equation is gives a rough estimate when the dome is approximately one-quarter of a sphere.

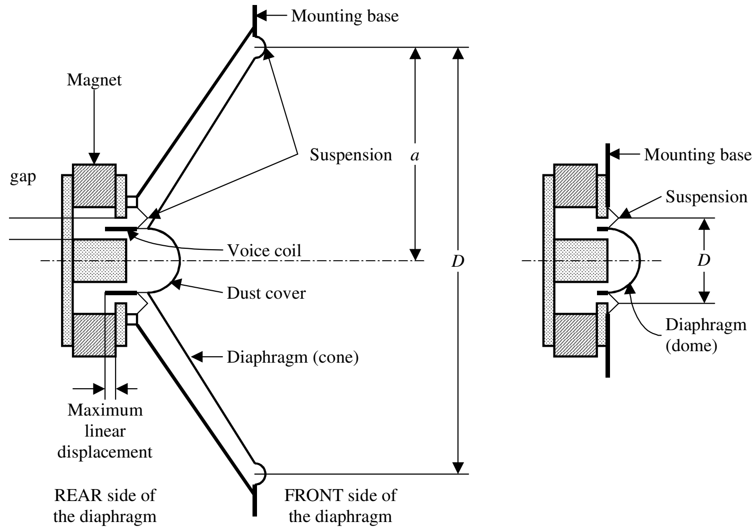


Figure 45 – Model of bass and midrange units (left) and treble units (right). The main difference between units is the mass of the moving system and the effective cross-sectional area of the diaphragm.

The loudspeaker will be assumed sealed to interrupt the radiation from the rear of the diaphragm thus reducing the complexity of the model. The intention is that the same model should apply to all loudspeaker units with adjustment of parameter values. Treble units are almost always sealed and midrange units most often include a can at the rear side. The bass loudspeakers are never sealed and must be placed within a closed cabinet.

As will be shown later in this section, the loudspeaker is a transducer where the transfer function is a band-pass filter with a frequency range determined by the moving system and the voice coil. The sound pressure is downward limited by the resonance frequency below which any closed-box loudspeaker systems will drop off at 12 dB/octave and the sound pressure is upward limited by the electrical low-pass filter of the voice coil resistance and inductance above which the response drops off by –6 dB/octave.

The model assumes that the loudspeaker diaphragm is vibrating as a rigid piston. This is true for low frequencies where the diaphragm circumference $2\pi a$ is short compared to wavelength λ. This is most often expressed as $ka < 1$, where frequency is represented by the angular wave number k:

$$ka < 1 \quad \wedge \quad k = \frac{\omega}{c} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda} \quad \Rightarrow \quad f < \frac{c}{2\pi a}$$

A loudspeaker with diameter a= 0.05 m becomes directive above approximately 1.1 kHz.

A loudspeaker can be assumed equivalent to a monopole sound source at low frequencies ($ka < 1$) where radiation is equally well in all directions. The loudspeaker becomes directive at high frequencies ($ka > 1$) where the output becomes concentrated on the loudspeaker axis and the off-axis output is limited. At higher frequencies ($ka > 3$) may the diaphragm break up and vibrate in sections with different phases, which affects both amplitude and directivity.

3.1.2. Electrical circuit

Electrical variables are the *voltage difference E* and *current I* and the electrical impedance is defined by:

$$Z_E = \frac{E}{I} \quad unit: \frac{V}{A} = \Omega$$

Input to the loudspeaker is an applied voltage E_G from an external voltage generator with an internal series impedance Z_G, see Figure 46. The resulting voice coil current I_C generates a voltage drop across the generator impedance of:

$$E_1 = Z_G I_C$$

The voice coil current flows through the voice coil resistance R_E and inductance L_E. The voice coil is shown in parallel with a resistor R_L to model the effect of eddy current losses in the magnetic circuit. The voltage across the voice coil is given by:

$$E_2 = \left(R_E + \frac{sL_E R_L}{sL_E + R_L} \right) \times I_C$$

When the voice coil and diaphragm is moving with velocity U_D, a voltage is induced into the voice coil wire:

$$E_M = BL \times U_D$$

The product of the magnet flux density B and the effective length of the voice coil wire L is the *force factor*, which is specified in the loudspeaker data sheet. We now know the individual terms of the electrical system.

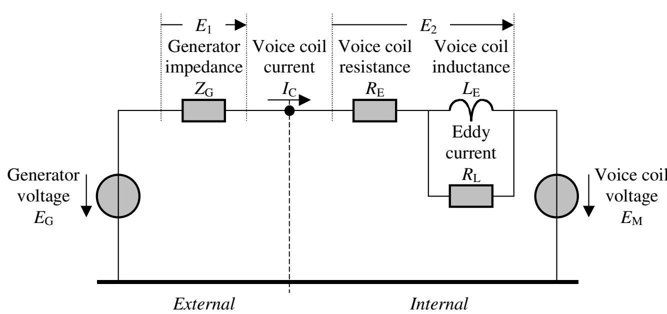


Figure 46 – Electrical model of loudspeaker input with the generator E_G representing the power amplifier and with the voice coil DC resistance R_E and the inductance L_E. The feedback from the moving system is represented by the second voltage source E_M.

The relation between generator voltage E_G, voice coil current I_C and diaphragm velocity U_D can now be derived by Kirchhoff's law, which states that the sum of all voltages within a closed mask must equal zero. It follows that:

$$E_G = \left(Z_G + R_E + \frac{sL_E R_L}{sL_E + R_L} \right) \times I_C + E_M$$

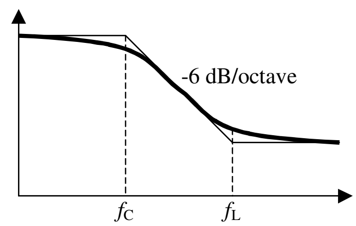
The generator impedance is often ignored since the output impedance approximates zero in most situations. This assumption will be used to simplify the expressions, but the Z_G can be re-introduced at any time by substituting R_E with $R_E + Z_G$.

Hence, the voice coil current becomes:

$$I_C = \frac{E_G - E_M}{R_E + \dfrac{sL_E R_L}{sLR_E + R_L}} = \frac{1 + \dfrac{sL_E}{R_L}}{1 + \dfrac{sL_E}{R_E}\left(1 + \dfrac{R_E}{R_L}\right)} \frac{E_G - E_M}{R_E}$$

The equation includes a filter with a cut-off frequency (pole frequency f_C) due to the voice coil series resistance R_E and inductance L_E and a correction due to the eddy current losses represented by R_L (null frequency f_L). The frequencies are:

$$\boxed{\begin{aligned} f_C &= \frac{\omega_C}{2\pi} = \frac{R_E\left(1 + \dfrac{R_E}{R_L}\right)}{2\pi L_E} \approx \frac{R_E}{2\pi L_E} \\ f_L &= \frac{\omega_L}{2\pi} = \frac{R_L}{2\pi L_E} \end{aligned}}$$



For R_E = 5 Ω and L_E = 1 mH is f_C = 800 Hz (ignoring R_L) so the higher frequencies are attenuated by –6 dB/octave. For R_L = 100 Ω is f_L = 16 kHz above which the attenuation ceases.

The voice coil current is represented by:

$$\boxed{I_C = H_{VC} \frac{E_G - E_M}{R_E}}$$

The transfer function H_{VC} due to the voice coil is defined by:

$$\boxed{H_{VC} = \frac{1 + \dfrac{s}{\omega_L}}{1 + \dfrac{s}{\omega_C}} = \frac{\omega_L + s}{\omega_C + s} = \frac{1 + s_L}{1 + s_C}}$$

For low frequencies is H_{VC} = 1, which indicates that the voice coil current I_C is proportional to the difference between the excitation voltage E_G and the velocity-induced voltage E_M and that the correlation constant is the DC resistance of the voice coil R_E. For high frequencies are the voice coil current reduced since the impedance of the inductance L_E is becoming the dominating term.

3.1.3. Mechanical circuit

Mechanical variables are *force F* and *velocity U*, which are analogue to voltage and current in electrical systems although the electrical current flows through series-connected components, whereas the velocity in mechanical systems is common to parallel-connected components. The mechanical impedance is defined by:

$$Z_M = \frac{F}{U} \quad unit: \frac{Ns}{m} = \frac{kg}{s}$$

The motor of the loudspeaker is the *voice coil*, which is located within the strong magnetic flux of the air gap. The electrical current I_C within the voice coil results in a mechanical force working on the voice coil:

$$F_M = BL \times I_C$$

A typical value of BL is 10 N/A so a voice coil current of one ampere results in a force on the voice coil of 10 N, which is approximating the weight of a 1 kg plumb.

A pressure difference between the front and rear side of the diaphragm results in a force working on the diaphragm, which is the pressure difference multiplied by the area:

$$F_A = S_D\left(p_F - p_R\right) = S_D p_D$$

For an 8 inch loudspeaker with a diaphragm area $S_D = 0.032\ \text{m}^2$, and a pressure difference of $p_D = 1$ Pa, corresponding to 94 dB sound pressure level, the force from the acoustical system becomes $f_A = 0.032$ N, which is small compared to the mechanical forces involved so the reaction from the acoustic system can be ignored in some applications.

The resultant force working upon the mechanical system becomes:

$$F_R = F_M - F_A$$

The direction of the forces has been selected so $F_M > 0$ drives the diaphragm forward while $F_A > 0$ drives it backward. This is an arbitrary choice, but it models the actual behaviour of the loudspeaker where the mechanical force is opposed by the force from the acoustical system.

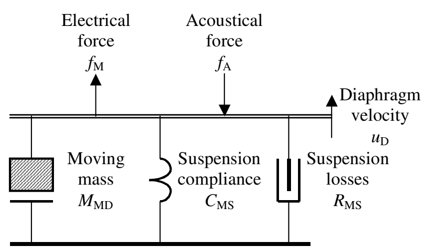


Figure 47 – Mechanical model of loudspeaker moving system (mobility analogy). The mass of voice coil and diaphragm are assembled into M_{MD}, the suspension system is represented by the spring compliance C_{MS} and the friction losses by R_{MS}.

Typical values for an 8 inch loudspeaker are: $M_{MD} = 80\ 10^{-3}$ kg, $C_{MS} = 500\ 10^{-6}$ m/N and $R_{MS} = 3$ kg/s.

The mechanical system consists of the mass M_{MD} of voice coil and diaphragm, and of the compliance C_{MS} and friction losses R_{MS} of the suspension system. The diaphragm velocity U_D is described as the solution to the following differential equation:

$$M_{MD}\frac{du_D}{dt} + R_{MS}u_D + \frac{1}{C_{MS}}\int u_D dt = F_M - F_A$$

The first term is due to the law of motion, $F_M = M_{MD}a_D$, where $a_D = du_D/dt$ is the acceleration of the diaphragm. The second term is the relation between force and friction losses, $F_R = R_{MS}u_D$. The third term is due to compliance, $F_C = x_D/C_{MS}$, where x_D is the diaphragm displacement and is related to velocity by $u_D = dx_D/dt$.

The mechanical system is completely described by the *resonance frequency* f_S and the *quality factor* Q_S.

$$f_S = \frac{1}{2\pi\sqrt{M_{MD}C_{MS}}}$$

$$Q_S = \frac{1}{R_{MS}}\sqrt{\frac{M_{MD}}{C_{MS}}}$$

In the frequency domain is the equation easy to solve for diaphragm velocity:

$$\left(sM_{MD} + R_{MS} + \frac{1}{sC_{MS}}\right)U_D = F_M - F_A$$

This can be written:

$$Z_M U_D = F_M - F_A \quad where \quad Z_M = sM_{MD} + R_{MS} + \frac{1}{sC_{MS}}$$

The corresponding circuit model is shown in Figure 48.

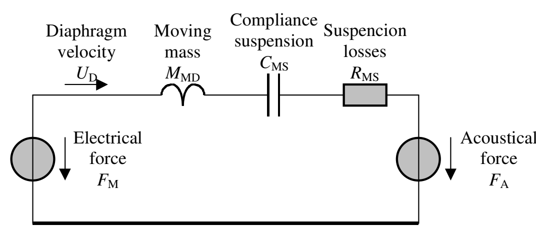


Figure 48 – Impedance analogy of the mechanical system. The mass and compliance cancels at the resonance frequency where the diaphragm velocity is maximum.

The diaphragm velocity is maximum around the resonance frequency where $Z_M \approx R_{MS}$ and is in phase with the excitation. Below the resonance frequency will the diaphragm velocity increase with increasing frequency and the phase approaches +90°, while the diaphragm velocity decreases with frequency above the resonance frequency where the phase approaches –90°, compared to the excitation.

3.1.4. Acoustical circuit

Acoustical variables are the *pressure difference p* and *volume velocity SU*, which are analogue to voltage and current in electrical systems. The volume velocity is in this study represented by the diaphragm cross-sectional area *S* multiplied by the velocity, hence the variable *SU*. The acoustical impedance is defined by:

$$Z_A = \frac{p}{SU} \quad unit: \frac{Ns}{m^3} = \frac{kg}{sm^2}$$

Two different load impedances will be analysed below; the loading at the front side of the diaphragm, which is assumed radiating into free space, and the loading at the rear side, which is radiating into a closed cabinet.

The front side of the diaphragm is loaded by air and the impedance is defined in [3] for a circular piston at the end of an infinitely long tube. The impedance consists of a mass term, which dominates for *ka* < 1, and a frequency-dependent resistive term, which dominates at higher frequencies and approximates a constant for $ka > 2$. The resulting impedance is a parallel combination of:

$$M_{AF} \approx 0.195 \frac{\rho_0}{a} \quad and \quad R_{AF} \approx 0.318 \frac{\rho_0 c}{a^2}$$

Examples are $M_{AF} = 2.3$ kg/m^4 and $R_{AF} = 20\ 10^3$ kg/sm^4 for $a = 0.1$ m (8 inch loudspeaker). The impedances are transformed to the mechanical circuit by multiplication with S_D^2. For $S_D = \pi a^2 = 30\ 10^{-3}$ m^2 the values become: $M_{MF} = 2.3$ g and $R_{MF} = 20\ 10^3$ kg/s.

The mass term represents a volume of uncompressed air, which is moving with the diaphragm velocity. The resistance term represents the energy lost into the air.

Since the components are in parallel the total impedance becomes:

$$Z_{AF} = \frac{sM_{AF} R_{AF}}{sM_{AF} + R_{AF}} = \frac{sM_{AF}}{1 + \frac{s}{\omega_{AF}}} \quad where \quad f_{AF} = \frac{\omega_{AF}}{2\pi} = \frac{R_{AF}}{2\pi M_{AF}}$$

The cut-off frequency f_{AF} is 1.4 kHz for the 8 inch loudspeaker.

A fair representation is a mass M_{AF} at low frequencies (LF) and a resistor R_{AF} at high frequencies (HF), which will be represented as follows (ka limits from [3]):

$$Z_{AF} = M_{AF}^{(LF)} \ for\ ka < 0.5 \quad and \quad Z_{AF} = R_{AF}^{(HF)} \ for\ ka > 5$$

The rear side of the loudspeaker is loaded by the closed cabinet, which is represented by an acoustical impedance with C_{AB} for the compliance of the confirmed air, M_{AB} for the air load on the rear side of the diaphragm and R_{AB} for the losses within the box. The acoustical loss within the closed box R_{AB} cannot easily be calculated and must be estimated by other means; but luckily, the value is of minor importance, at least for this study, and can safely be ignored. The compliance and mass load can be calculated by the following approximations, which assumes that the box is small and without any damping material.

$$C_{AB} \approx \frac{V_{AB}}{\rho_0 c^2} \quad and \quad M_{AB} \approx 0.65 \frac{\rho_0}{\pi a}$$

A box is small when the largest dimension is less than $\lambda/10$, which corresponds to the following design requirement where L_B represents the largest box dimension:

$$f > \frac{c}{10 L_B}$$

The equivalent circuit for the loading of the loudspeaker is shown in Figure 49, with the rear of the loudspeaker radiating into a closed box and the front into free space.

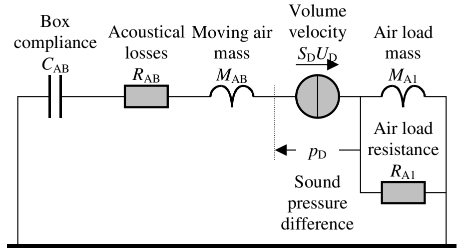


Figure 49 – Load impedances for the loudspeaker.

Hence, the sound pressure difference between the loudspeaker front and rear side:

$$p_D = \left(sM_{AF}^{(LF)} + R_{AF}^{(HF)} + sM_{AB} + R_{AB} + \frac{1}{sC_{AB}} \right) S_D U_D = (Z_{AF} + Z_{AR}) S_D U_D$$

Radiation resistance Z_{AF} is a function on loudspeaker mounting. The loudspeaker may be approximated by a monopole sound source at low frequencies but the loudspeaker baffle comes into play at higher frequencies and the model is then approximately a monopole close to an infinite wall, which doubles the sound pressure. The transition between the two models is unfortunately not straightforward.

The model should also include the interaction between two sources operating at the same frequency (i.e. at cross-over) and the reflections from cabinet boundary (diffraction). In order to separate the different models, the loudspeaker model will load the front of the loudspeaker by the radiation impedance of a plane piston in an infinite tube. This models the monopole sound source and the other factors can then be added to the model without a need of changing the basic model.

3.1.5. Diaphragm velocity

The first step is to derive a relation between the diaphragm velocity U_D and generator voltage E_G and the final step is to determine the sound pressure.

Two expressions are at hand for the voice coil current I_C and using them to eliminate the voice coil current introduces E_G, U_D and the force F_M from the voice coil.

$$I_C = H_{VC} \times \frac{E_G - BLU_D}{R_E} = \frac{F_M}{BL} \quad \Rightarrow \quad F_M = H_{VC} \times \frac{BLE_G - (BL)^2 U_D}{R_E}$$

H_{VC} represents the voice coil high-frequency attenuation, which is unity for frequencies below approximately 500 Hz.

Another expression for F_M is available introducing the mechanical impedance Z_M and the force from the acoustical system F_A, which again introduces the acoustical impedances Z_{AF} and Z_{AR} for the loading on the loudspeaker front and rear sides.

$$F_M = Z_M U_D + F_A \quad \Rightarrow \quad F_M = Z_M U_D + (Z_{AF} + Z_{AR}) S_D^2 U_D$$

Elimination of F_M results in the following equation for estimation of diaphragm velocity U_D versus generator voltage E_G.

$$H_{VC} \times \frac{BLE_G - (BL)^2 U_D}{R_E} = U_D Z_M + (Z_{AF} + Z_{AR}) S_D^2 U_D$$

The expression is solved for diaphragm velocity U_D as a function of E_G:

$$U_D = \frac{1}{\frac{(BL)^2 H_{VC}}{R_E} + Z_M + (Z_{AF} + Z_{AR})S_D^2} \times H_{VC} \times \frac{BL}{R_E} \times E_G$$

The first term is consists of mechanical impedances with the voice coil H_{VC} transformed to the mechanical system by $(BL)^2$, and the acoustical impedances transformed by S_D^2.

The $(BL)^2 H_{VC}/R_E$ term represents the damping from the electrical system. The term is active around resonance where the denominator consists of $(BL)^2/R_E$ plus the frequency-independent terms from Z_M, Z_{AF} and Z_{AR} (the reactive terms cancel at resonance). The $(BL)^2/R_E$ term is gradually removed at high frequencies due to H_{VC}, but the denominator is dominated by the mass-term of Z_M above resonance, so H_{VC} can be ignored without consequences at higher frequencies. This can be seen from the following inequity, which states the requirement for the mass term to dominate over the $(BL)^2/R_E$ term:

$$\frac{(BL)^2}{R_E} < |sM_{MD}| = \omega M_{MD} \Rightarrow f > \frac{(BL)^2}{2\pi R_E M_{MD}}$$

For BL = 10 N/A, R_E = 5 Ω and M_{MD} = 0.050 kg is f >160 Hz. The H_{VC} term is unity for frequencies below approximately 500 Hz, so the correction due to H_{VC} can be ignored.

Using this simplification and introducing the definitions for the mechanical impedance Z_M and the radiation resistance Z_{AF} and Z_{AR} the equation becomes.

$$U_D = \frac{H_{VC} \times \frac{BL}{R_E} \times E_G}{\frac{(BL)^2}{R_E} + sM_{MD} + R_{MS} + \frac{1}{sC_{MS}} + \left(sM_{AF}^{(LF)} + R_{AF}^{(HF)} + sM_{AB} + R_{AB} + \frac{1}{sC_{AB}} \right) S_D^2}$$

As stated before, the acoustical impedance for the front of the diaphragm consists of two contributions, the mass M_{AF}, which is effective at low frequency (ka < 0.5), and the resistance R_{AF}, which is effective at high frequency (ka > 5). Note that the two terms are *not* active at the same time, hence the notation with super fixes LF and HF.

As will be seen in the following, the resistance term R_{AF} is of no importance for the definition of the resonance frequency, since the resonance frequency this is a low-frequency parameter, and it will be completely masked by the mass of the moving system at high frequencies, hence R_{AF} will never contribute to the transfer function and will be ignored.

The above simplifications do not limit the useful frequency range of the model; we are simply deleting terms, which will never contribute to the result. The model is thus valid from the extreme low-frequency range and up to the limit where the loudspeaker diaphragm breaks up, i.e. about around ka = 3.

The following definitions will be introduced to ease the analysis: M_{MS}, which is the moving mass of the total system, R_{MT}, which is the mechanical resistance of the total and C_{MT}, which is the mechanical compliance of the total system:

$$\boxed{\begin{aligned} &M_{MS} = M_{MD} + (M_{AF} + M_{AB})S_D^2 \\ &R_{MT} = \frac{(BL)^2}{R_E} + R_{MS} + R_{AB}S_D^2 \\ &\frac{1}{sC_{MT}} = \frac{1}{sC_{MS}} + \frac{S_D^2}{sC_{AB}} \Rightarrow C_{MT} = \frac{C_{MS}C_{AB}}{S_D^2 C_{MS} + C_{AB}} = \frac{C_{MS}\dfrac{C_{AB}}{S_D^2}}{C_{MS} + \dfrac{C_{AB}}{S_D^2}} \end{aligned}}$$

Hence, the following equation for the diaphragm velocity:

$$U_D = \frac{1}{sM_{MS} + R_{MT} + \dfrac{1}{sC_{MT}}} \times H_{VC} \times \frac{BL}{R_E} \times E_G$$

The equation is multiplied and divided by R_{MT} and the nominator and denominator are multiplied by sC_{MT} thus resulting in a second-order denominator polynomial in s.

$$U_D = \frac{sR_{MT}C_{MT}}{s^2 M_{MS}C_{MT} + sR_{MS}C_{MT} + 1} \times H_{VC} \times \frac{BL}{R_{MT}R_E} \times E_G$$

The resonance frequency f_S is identified as the frequency where $s^2 M_{MS} C_{MT} + 1 = 0$:

$$\boxed{f_S = \frac{\omega_S}{2\pi} = \frac{1}{2\pi\sqrt{M_{MS}C_{MT}}}}$$

The second-order term is normalised to $(s/\omega_S)^2$ and the first-order term $sR_{MS}C_{MT}$ is normalised as follows:

$$sR_{MT}C_{MT} = \frac{s}{\omega_S}\omega_S R_{MT}C_{MT} = \frac{s}{\omega_S}\frac{R_{MT}C_{MT}}{\sqrt{M_{MS}C_{MT}}} = \frac{s}{\omega_S}R_{MT}\sqrt{\frac{C_{MT}}{M_{MS}}}$$

The constant factor to s/ω_S is dimension-less and is identified as the *total quality factor* for the loudspeaker in a closed cabinet:

$$\boxed{Q_{TS} = \frac{1}{R_{MT}}\sqrt{\frac{M_{MS}}{C_{MT}}}}$$

The total quality factor is sensitive to both the electrical, mechanical and acoustical systems but damping from the electrical system is the dominating factor.

$Q_{TS} = 0.50$ corresponds to a *critically damped* system, which should theoretically reproduce transients without oscillations. The level is 6 dB down at the resonance frequency, which can be compensated electronically by a bass boost from the amplifier.

$Q_{TS} = 0.71$ corresponds to a *Butterworth filter* with –3 dB at the resonance frequency and some oscillations with transient input. It is often called the *maximally flat alignment*.

Q_{TS} = 1.00 corresponds to a *1 dB Chebychev* filter with 0 dB at the resonance frequency, a peaking of 1 dB above the resonance frequency and oscillations with transients.

It is common to divide the quality factor into electrical and mechanical quality factors by the use of the definition for R_{MT} and ignoring the acoustical resistance, which is small compared to the two others.

$$Q_{ES} = \frac{R_E}{(BL)^2}\sqrt{\frac{M_{MS}}{C_{MS}}} \quad Q_{MS} = \frac{1}{R_{MS}}\sqrt{\frac{M_{MS}}{C_{MS}}} \quad Q_{TS} = \frac{Q_{ES}Q_{MS}}{Q_{ES}+Q_{MS}}$$

The quality factor is proportional to R_E. This reflects the recommendation, found in many loudspeaker books and magazines, that a bass loudspeaker should not be driven from a high-impedance source. This would otherwise increase the effective value of R_E (which is in series with Z_G) and result in a booming bass.

The transfer function for diaphragm velocity U_D as a function of generator voltage E_G can then be expressed as follows:

$$\boxed{U_D = \frac{\frac{1}{Q_{TS}}\frac{s}{\omega_S}}{\left(\frac{s}{\omega_S}\right)^2 + \frac{1}{Q_{TS}}\frac{s}{\omega_S} + 1} \times H_{VC} \times \frac{BL}{R_{MT}R_E} \times E_G}$$

The expression for the diaphragm velocity is a *band-pass filter* centred at the resonance frequency and with symmetrical skirts. The diaphragm velocity is maximum at the resonance frequency and less for all other frequencies.

3.1.6. Sound pressure

Derivation of the sound pressure from the loudspeaker uses the equation for the sound pressure at distance r from a monopole sound source in free space. The sound pressure is specified with a given volume velocity $S_D U_D$ and is [3]:

$$p(r) = \frac{i\omega\rho_0}{4\pi r}\exp(-ikr)S_D U_D$$

The exponential is a complex unit vector which defines the phase due to the delay of the sound waves travelling the distance r from the source to the monitoring point where k is the angular wave number ($k = \omega/c$).

The sound pressure becomes:

$$p(r) = \frac{i\omega\rho_0}{4\pi r}\exp(-ikr)S_D \times \frac{\frac{1}{Q_{TS}}\frac{s}{\omega_S}}{\left(\frac{s}{\omega_S}\right)^2 + \frac{1}{Q_{TS}}\frac{s}{\omega_S} + 1} \times H_{VC} \times \frac{BL}{R_{MT}R_E} \times E_G$$

Which can be rearranged into five terms: two transfer functions, one angular transfer function, a constant expression and the excitation voltage:

$$p(r) = H_{VC} \times \frac{\frac{i\omega s}{\omega_S^2}}{\left(\frac{s}{\omega_S}\right)^2 + \frac{1}{Q_{TS}}\frac{s}{\omega_S} + 1} \times \exp(-ikr) \times \frac{\rho_0 \omega_S S_D BL}{4\pi r Q_{TS} R_{MT} R_E} \times E_G$$

The resonance frequency has been included into the constant expression in order to simplify the expression.

Using the relation $s = i\omega$ and introducing a reference distance r_{REF}, the resultant transfer function for the sound pressure $p(r)$ at distance r given by the excitation voltage E_G:

$$\boxed{p(r) = H_{VC} \times H_D \times \frac{r_{REF}}{r} \exp(-ikr) \times KE_G}$$

The transfer function due to the mechanical system is:

$$\boxed{H_D = \frac{\left(\frac{s}{\omega_S}\right)^2}{\left(\frac{s}{\omega_S}\right)^2 + \frac{1}{Q_{TS}}\frac{s}{\omega_S} + 1}}$$

The constant K assembles the scaling factors:

$$K = \frac{\rho_0 f_S S_D BL}{2 r_{REF} Q_{TS} R_{MT} R_E}$$

Expressing R_{MT} by the definition of the total quality factor and substituting C_{MT} by the definition of the resonance frequency, the mechanical losses can be expressed as:

$$R_{MT} = \frac{1}{Q_{TS}}\sqrt{\frac{M_{MS}}{C_{MT}}} = \frac{1}{Q_{TS}}\sqrt{\frac{M_{MS}}{\frac{1}{\omega_S^2 M_{MS}}}} = \frac{M_{MS}\omega_S}{Q_{TS}}$$

The scaling factor is thus:

$$\boxed{K = \frac{\rho_0 S_D BL}{4\pi r_{REF} M_{MS} R_E}}$$

The loudspeaker is *mass-controlled* above the resonance frequency, i.e. the diaphragm movement is controlled almost entirely by the force from the voice coil and the mass of the moving system M_{MS}. This represents the normal use of the loudspeaker. The scaling factor is valid for all frequencies but assumes a diaphragm vibrating as a rigid piston. The effective frequency limit is thus approximately $ka < 3$:

$$\boxed{f_{MAX} \approx \frac{3c}{2\pi a}}$$

For an 8 inch loudspeaker with $a = 0.1$ m the upper limit is approximately 1.6 kHz.

3.2. Loudspeaker pass band

Loudspeakers are band-pass filters with a transfer function given by $H_{VC}H_D$ from the previous section. The amplitude response is shown in Figure 50 for a theoretical loudspeaker with the frequency axis normalised to the resonance frequency.

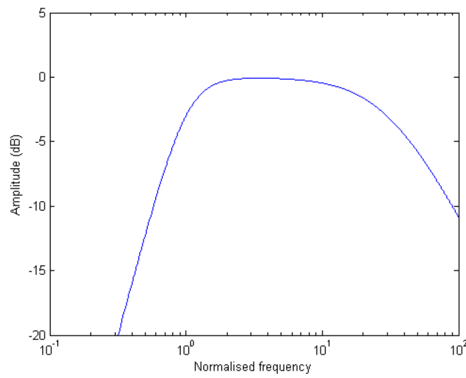


Figure 50 – Response of a loudspeaker with system resonance frequency normalised to unity, total quality factor Q_{TC} = 0.71 and the voice-coil frequency at 30 times the system resonance.

The useful *pass band* is the frequency range above the mechanical resonance frequency (normalised to unity) and below the cut-off due to the voice coil inductance (arbitrarily set to 30 times the resonance frequency). Frequencies below the mechanical resonance frequency are attenuated 12 dB/octave and high frequencies are attenuated –6 dB/octave due to the voice coil inductance.

The actual performance may depart from the typical view shown above due to the construction affecting the voice coil impedance, the loudspeaker directivity and diaphragm break-up.

3.2.1. Sound pressure level

The sound pressure level is calculated from:

$$L(r) = 20\log_{10}\left(\frac{|p(r)|}{p_{REF}}\right) dB$$

The reference sound pressure is p_{REF} = 20 10^{-6} Pa and corresponds to 0 dB of sound pressure level, i.e. the threshold of hearing around 1 kHz.

The *reference sound pressure level* L_{REF}, at distance r_{REF} (typically 1 m) and excitation voltage E_{REF} (typically 2.83 V for 1 W of power dissipated into 8 Ω) and without the influence from a loudspeaker baffle (radiating into 4π), is:

$$\boxed{L_{REF(4\pi)} = 20\log_{10}\left(\frac{\rho_0 S_D BL}{4\pi r_{REF}\, p_{REF} M_{MS} R_E} E_{REF}\right) dB}$$

The reference sound pressure level is used to characterise the loudspeaker sensitivity and is most often measured with radiation into 2π. In order to compare the calculation to the measurement the formula can be changed to radiation into 2 π:

$$\boxed{L_{REF(2\pi)} = 20\log_{10}\left(\frac{\rho_0 S_D BL}{2\pi r_{REF}\, p_{REF} M_{MS} R_E} E_{REF}\right) dB}$$

Measurements on the loudspeaker typically use a rigid plane that forms one of the walls of the anechoic chamber. This is a practical implementation of an infinite baffle.

Example 1. Peerless 315SWR: $S_D = 50\ 10^{-3}$ m^2, $BL = 11.6$ N/A, $R_E = 5.5\ \Omega$, $M_{MS} = 84\ 10^{-3}$ kg, and $E_{REF} = 2.83$ V. The sensitivity at $r_{REF} = 1$ m becomes 90.6 dB re. 20 µPa. The specification is $L_{REF} = 89.3$ dB, which is 1.3 dB below the calculated.

Example 2. Peerless 205WR33: $S_D = 22\ 10^{-3}$ m^2, $BL = 9.6$ N/A, $R_E = 6.0\ \Omega$, $M_{MS} = 26\ 10^{-3}$ kg, and $E_{REF} = 2.83$ V. The sensitivity at $r_{REF} = 1$ m becomes 91.3 dB re. 20 µPa. The specification is $L_{REF} = 90$ dB, which is 1.3 dB below the calculated.

3.2.2. Diaphragm excursion

Loudspeaker design requires observation of the diaphragm excursion in order to design a loudspeaker that can withstand the intended use. An important parameter is the diaphragm excursion, since the system may depart from the assumption of linearity at high levels and low frequencies.

The analysis starts from the previously derived expression for diaphragm velocity:

$$U_D = H_{VC} \times \frac{\dfrac{1}{Q_{TS}}\dfrac{s}{\omega_S}}{\left(\dfrac{s}{\omega_S}\right)^2 + \dfrac{1}{Q_{TS}}\dfrac{s}{\omega_S} + 1} \times \frac{BL}{R_{MT} R_E} \times E_G$$

The second-order function is a band-pass filter with a maximum at the system resonance frequency f_S, so the diaphragm velocity is proportional to frequency below resonance and inversely proportional to frequency above resonance. The loudspeaker is assumed used above the resonance frequency and since the following analysis will deal with low frequencies can the first term be assumed unity.

A good approximation for the diaphragm velocity above the resonance frequency and below the voice coil cut-off frequency is obtained by observing that the $(s/\omega)^2$ term dominates and the H_{VC} is unity. The diaphragm velocity can thus be described by:

$$U_D \xrightarrow[\omega<\omega_C]{\omega>\omega_S} U_D^{\circ} = \frac{1}{Q_{TS}}\frac{\omega_S}{s} \times \frac{BL}{R_{MT} R_E} \times E_G$$

The equation can be rewritten with the aid of the definition of Q_{TS}:

$$U_D^{\circ} = \frac{1}{Q_{TS}}\frac{\omega_S}{s} \times \frac{BL}{R_{MT} R_E} \times E_G = \frac{BL}{s M_{MS} R_E} \times E_G$$

Diaphragm excursion can be calculated from:

$$x_D(t) = \int U_D(t)\,dt \quad \xleftrightarrow{\text{Frequency transformation}} \quad x_D(s) = \frac{U_D(s)}{i\omega}$$

Hence, and ignoring the minus sign from $(i\omega)^2$:

$$x_D = \frac{BL}{\omega^2 M_{MS} R_E} \times E_G$$

The lowest allowable frequency (above resonance frequency) for a maximum excursion x_D and generator voltage E_G is:

$$\boxed{f_{MIN} = \frac{1}{2\pi}\sqrt{\frac{BL \times E_G}{x_D M_{MS} R_E}}}$$

Example. For BL = 10 N/A, E_G = 40 V ($E_{G\,RMS}$ = 28.3 V for 100 W into 8 Ω), x_D = 5 mm, M_{MS} = 30 g and R_E = 6 Ω the limit is 106 Hz. With E_G = 4 V (for 1 W into 8 Ω) the same loudspeaker can operate down to 35 Hz.

For sinusoidal excitation at x_D excursion the oscillation is described as:

$$x(t) = \mathrm{Re}\{x_D \exp(i\omega t)\}$$

The maximum diaphragm velocity is found as the maximum of the derivative of the oscillation:

$$u_{MAX} = \left.\frac{dx(t)}{dt}\right|_{MAX} = \mathrm{Re}\{x_D i\omega \exp(i\omega t)\}\Big|_{MAX} = \omega x_D$$

The maximum free field sound pressure becomes:

$$p(r) = \frac{i\omega\rho_0}{4\pi r}\exp(-ikr) S_D \omega x_D = \frac{i\pi\rho_0 f^2 S_D x_D}{r}\exp(-ikr)$$

The maximum sound pressure level at frequency f and at the reference distance r_{REF} and with an excursion of x_D is:

$$\boxed{L_{MAX\,(4\pi)} = 20\log_{10}\left(\frac{\pi\rho_0 f^2 S_D x_D}{r_{REF}\, p_{REF}}\right) dB}$$

Example. For an 8 inch loudspeaker with S_D = 30 10^{-3} m^2 and x_D = 5 mm excursion will the maximum sound pressure level at r_{REF} = 1 m and f = 35 Hz be L_{MAX} = 91 dB. A 12 inch loudspeaker with S_D = 60 10^{-3} m^2 and same linear excursion would produce L_{MAX} = 97 dB.

3.2.3. SPICE simulation model

Based upon the previous sections, the loudspeaker simulation model becomes as shown in Figure 51.

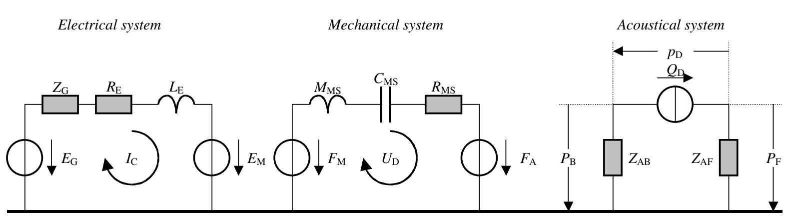


Figure 51 – Loudspeaker model with input from the generator (E_G) at the left side and output at the front side of the diaphragm (p_F) at the right side.

3.3. Directivity

A loudspeaker does not distribute the radiation evenly across space although it is often assumed to do so at low frequencies in order to simplify things. As diaphragm diameter or frequency increases, the loudspeaker concentrates the radiation on-axis.

The sound pressure at distance r for a circular piston with radius a and volume velocity $Q = S_D U_D$ in an infinite baffle is [3]:

$$p(r,\theta)=\frac{i\omega\rho Q}{2\pi r}\left[\frac{2J_1(ka\sin(\theta))}{ka\sin(\theta)}\right]\exp(-ikr)$$

The expression within the square brackets represents the directivity. The on-axis response is obtained for $ka\sin(\theta)$ small where the Bessel function can be approximated:

$$J_1(ka\sin(\theta))=\frac{ka\sin(\theta)}{2}$$

Using this approximation the on-axis sound pressure becomes:

$$p(r,\theta)=\frac{i\omega\rho Q}{2\pi r}\exp(-ikr)$$

This shows that the far-field on-axis pressure radiated by a circular disk in an infinite baffle is identical to the pressure radiated by a point source against a rigid surface.

The conclusion is that the directivity of a loudspeaker in a cabinet can be modelled by:

$$\boxed{D(\theta)=\frac{2J_1(ka\sin(\theta))}{ka\sin(\theta)}}$$

The directivity is plotted in Figure 52. Low frequencies ($ka < 1$) are virtually unaffected by the observation angle (less than 1 dB attenuation) while high frequencies ($ka > 1$) are attenuated for the off-axis radiation.

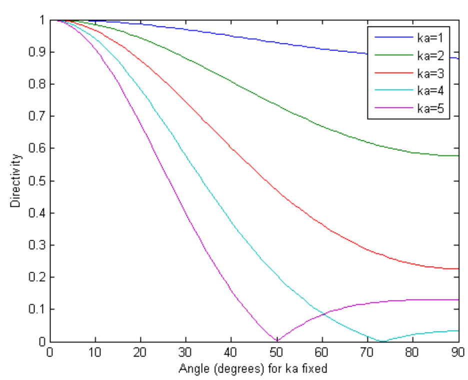


Figure 52 – Directivity for a loudspeaker in an infinite baffle at four different values of *ka*, where *a* is diaphragm radius.

The directivity is of minor importance for $ka > 3$ since the loudspeaker may suffer from diaphragm break-up, which drastically changes both amplitude and directivity, and this behaviour is hard to model.

3.4. Diffraction

Loudspeaker units are in this report assumed mounted within one of the walls of a closed cabinet so the proximity of the loudspeaker is a rigid surface. The cabinet dimensions can be considered small only at low frequencies where the loudspeaker is radiating into full space (4π solid angle). Cabinet dimensions become comparable to wavelength at increasing frequency and the radiation of high-frequency waves become restricted to the half-space (2π solid angle) in front of the cabinet thus increasing the sound pressure by 6 dB. The increase in sound pressure toward high frequencies is known as diffraction and will be introduced.

3.4.1. Circular baffle

It is assumed that the direct signal from the loudspeaker is radiated from a point source into half space in front of the loudspeaker cabinet (p_D, blue colour in Figure 53). At the edge of the cabinet is some part of the wave propagated along the side of the cabinet in order to fill the area behind the cabinet (p_B, green). The discontinuity at the edge gives rise to a reflection of opposite polarity (p_R, red). It will, for the case of simplicity, be assumed that the sound pressure is halved at the discontinuity so the backward radiation and the forward reflection are of equal amplitude but opposite polarity.

The effect of cabinet thickness will not be considered.

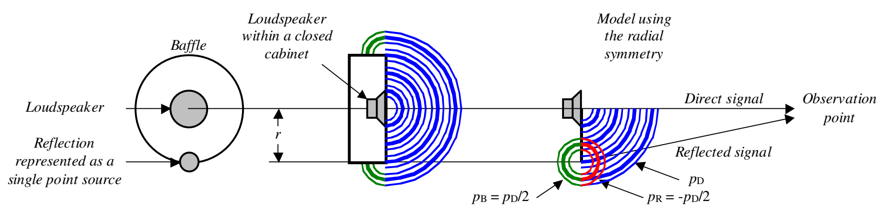


Figure 53 – A simplified model assuming the loudspeaker located at a circular baffle with negligible thickness (closed cabinet). The reflections are assumed assembled into one point source located at the edge of the surface.

The sound pressure at observation distance R from the loudspeaker is represented by a point source with volume velocity $Q = S_D U_D$, which is radiating into the half space at the front of the loudspeaker:

$$p_D(R) = \frac{i\omega Q}{2\pi R}\exp(-ikR)$$

Here is $k = \omega/c = 2\pi f/c$ the angular wave number.

Reflection from the cabinet boundary is delayed by distance r, the radius of the baffle, the sound pressure is one-half that of the loudspeaker sound pressure and the polarity is the opposite in order to partially cancel the radiation from the loudspeaker outside the baffle. The attenuation due to the increased path length will be ignored (distance r is assumed small compared to the observation distance R), and the reflected sound pressure at the observation point becomes:

$$p_R(R) = -\frac{i\omega Q}{4\pi R}\exp(-ik(R+r))$$

The resulting far field sound pressure on the axis of symmetry is the sum of the two expressions:

$$p_F(R) = \frac{i\omega Q}{2\pi R}\exp(-ikR) - \frac{i\omega Q}{4\pi R}\exp(-ik(R+r))$$

This can be reduced to:

$$p_F(R) = \frac{i\omega Q}{2\pi R}\exp(-ikR)\left[1 - \frac{1}{2}\exp(-ikr)\right]$$

The first two terms are identified as p_D, the direct sound from the loudspeaker, so the far field sound pressure with a circular baffle can be described as:

$$\boxed{p_F(R) = \left[1 - \frac{1}{2}\exp(-ikr)\right]p_D(R)}$$

At low frequencies, where the exponential approaches unity, is the level one-half that of a loudspeaker within an infinite baffle, i.e. the level is 6 dB down compared to the specification in the data sheet (measured at 2π). At higher frequencies, where the exponential rotates within the unit circle, is the amplitude oscillating from 0.5 to 1.5 times the level with an infinite baffle (from –6 dB to 3.5 dB).

Nominal level (0.0 dB)	$\left[1 - \frac{1}{2}\exp(-ikr)\right] = 1$	$kr = \arccos\left(\frac{1}{4}\right) \approx 1.32$	$f_{NOM} = 0.21\frac{c}{r}$
First peak (+3.5 dB)	$\exp(-ikr) = 1$	$kr = \pi \approx 3.14$	$f_{PEAK} = 0.50\frac{c}{r}$
First dip (-6.0 dB)	$\exp(-ikr) = -1$	$kr = 2\pi \approx 6.28$	$f_{DIP} = \frac{c}{r}$

For a circular baffle with radius r= 0.1 m, the nominal level is crossed first time at 720 Hz, the first peak is at 1.7 kHz and the first dip is at 3.4 kHz.

The behaviour of the equation within the square brackets is shown in Figure 54. The ripple at higher frequencies will be less pronounced in real life since the loudspeaker become directive thus reducing the effect of the reflection.

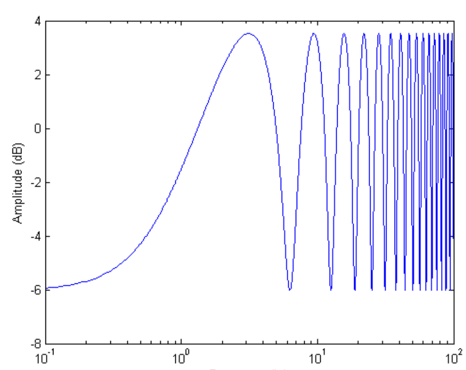


Figure 54 – Diffraction for a loudspeaker located at the centre of a circular tube of radius *r*. The model does not account for loudspeaker directivity or baffle thickness.

3.4.2. Sectional baffle

Assume a loudspeaker baffle designed from sections of circles, such as the one in Figure 55, where four sections of 90° each are shown. The number of sections N is arbitrary, as are the radii r_k of the sections, but the analysis assumes that the sections are all covering the same angle ($\varphi_k = 360°/N$).

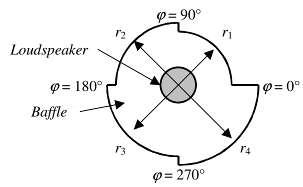


Figure 55 – Loudspeaker baffle constructed of four circular sections in order to smear the amplitude ripple.

The sound pressure of the direct signal from the loudspeaker is as before modelled by a point source radiating into half space:

$$p_D(R) = \frac{i\omega Q}{2\pi R}\exp(-ikR)$$

The reflections are modelled by point sources, one for each section, each with a sound pressure of $1/2N$ of the sound pressure of the direct signal and of opposite polarity.

$$p_n(R) = -\frac{i\omega Q}{4N\pi R}\exp(-ik(R + r_n)) \quad n = 1,2,\ldots,N$$

The resulting far field sound pressure becomes:

$$p_F(R) = p_D(R) + \sum_{n=1}^{N} p_n(R)$$

Inserting the expressions and collecting terms:

$$p_F(R) = \frac{i\omega Q}{2\pi R}\exp(-ikR)\left[1 - \frac{1}{2N}\sum_{n=1}^{N}\exp(-ikr_n)\right]$$

This is equivalent to:

$$\boxed{p_F(R) = \left[1 - \frac{1}{2N}\sum_{n=1}^{N}\exp(-ikr_n)\right]p_D(R)}$$

At low frequencies are all exponentials approximating unity so the sum equals N and the expression within the square brackets becomes one-half, so the initial level is –6 dB compared to the infinite baffle.

An example is shown in Figure 56 with four different values of the radii. The peak-peak ripple is reduced from 9 dB for the circular baffle to 2 dB for $kr < 15$ by this set-up.

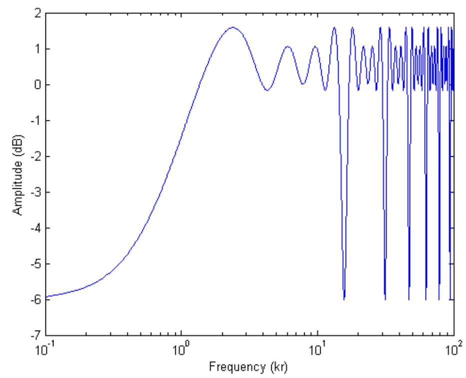


Figure 56 – Diffraction for a baffle consisting of four circular sections, each 90° wide and with radii r_1:r_2:r_3:r_4 = 0.4:0.8:1.2:1.6, which scales the result so it is comparable to the previous plot.

3.4.3. Square baffle

A square baffle is shown in Figure 57, and differs from the previous analysis by the distance from loudspeaker to edge being a function of angle φ. The symmetry of the system can simplify the analysis since there are eight identical sections, which can be represented by the section from 0 to $\pi/4$ and the result can be repeated eight times.

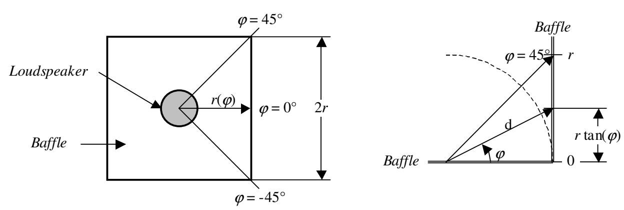


Figure 57 – A square baffle with side length 2*r* being defined by polar coordinates. Symmetry divides the baffle into eight identical sections of 45° each.

The distance from the centre of the loudspeaker to the edge is given by Pythagoras as:

$$d^2 = r^2 + (r\cos(\varphi))^2$$

Using $\sin^2\varphi + \cos^2\varphi = 1$, this can be reduced to:

$$d(\varphi) = \frac{r}{\cos(\varphi)} = r\sec(\varphi)$$

The free-field radiation from each of the eight sections at distance R become:

$$p_n(R) = -\frac{1}{8}\frac{i\omega Q}{4\pi R}\int_0^{\pi/4} \exp[-ikr\sec(\varphi)]d\varphi, \quad n = 1,2,\ldots,8$$

Factor one-eights is due to symmetry and the reduced range of the angle. The secant can be approximated by the Taylor series expansion [*Schaum* 20.25]:

$$\sec(\varphi) = 1 + \frac{\varphi^2}{2!} + \frac{5\varphi^4}{24} + \frac{61\varphi^6}{720} + \ldots, \quad |\varphi| < \frac{\pi}{2}$$

It is possible to terminate the series after the second term since the third term is 0.08 for $\varphi = \pi/4$ and the higher-order terms are even smaller.

The sound pressure becomes:

$$p_n(R) = -\frac{i\omega Q}{32\pi R}\int_0^{\pi/4} \exp\left[-ikr\left(1+\frac{\varphi^2}{2}\right)\right]d\varphi$$
$$= -\frac{i\omega Q}{32\pi R}\exp(-ikr)\int_0^{\pi/4}\exp\left(-ikr\frac{\varphi^2}{2}\right)d\varphi$$

The Taylor series expansion of the exponential is [*Schaum* 20.15]:

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \quad |x| < \infty$$

The series expansion converges for any x if the number of terms is sufficiently large so the expansion is not an approximation but the series will be terminated after integration has been carried out. The integral becomes:

$$\int_0^{\pi/4}\exp\left(\frac{-ikr\varphi^2}{2}\right)d\varphi = \int_0^{\pi/4}\left[1+\frac{-ikr\varphi^2}{2}+\frac{1}{2!}\left(\frac{-ikr\varphi^2}{2}\right)^2+\frac{1}{3!}\left(\frac{-ikr\varphi^2}{2}\right)^3+\ldots\right]d\varphi$$
$$= \int_0^{\pi/4}\left[1-i\frac{kr}{2}\varphi^2-\frac{(kr)^2}{8}\varphi^4+i\frac{(kr)^3}{48}\varphi^6+\ldots\right]d\varphi$$
$$= \left[\varphi-i\frac{kr}{6}\varphi^3-\frac{(kr)^2}{40}\varphi^5+i\frac{(kr)^3}{336}\varphi^7+\ldots\right]_0^{\pi/4}$$
$$= \frac{\pi}{4}-i\frac{kr}{6}\left(\frac{\pi}{4}\right)^3-\frac{(kr)^2}{40}\left(\frac{\pi}{4}\right)^5+i\frac{(kr)^3}{336}\left(\frac{\pi}{4}\right)^7+\ldots$$
$$= 0.7854 - i0.0807kr - 0.0075(kr)^2 + i0.0005(kr)^3 + \ldots$$
$$\approx 0.79 - i0.15kr$$

The approximation is valid for vanishing higher-order terms. This requires that:

$$\frac{0.0075(kr)^2}{0.7854} < 1 \quad \Rightarrow \quad kr < \sqrt{\frac{0.7854}{0.0075}} = 10 \quad \Rightarrow \quad f < \frac{5c}{\pi r}$$

For $r = 0.15$ m (a baffle width of 0.3 m) the limiting frequency becomes $f < 3.6$ kHz.

The approximation is a phase vector with an amplitude of 0.79 and a phase of 0° at low frequencies and an amplitude of 0.80 and a phase of 11° at $kL = 1$. The change in amplitude can be neglected and the phase becomes:

$$\arctan\left(\frac{-0.15kr}{0.79}\right) = -\arctan(0.19kr)$$

Hence, the integral can be represented by the approximation:

$$\int_0^{\pi/4} \exp\left(\frac{-ikr\varphi^2}{2}\right) d\varphi \approx 0.79\exp(-0.19ikr)$$

The sound pressure from one of the edges becomes:

$$p_k(R) = -\frac{i\omega Q}{32\pi R}\exp(-ikr)0.79\exp(-0.19ikr)$$
$$= -0.0494\frac{i\omega Q}{2\pi R}\exp(-1.19ikr)$$

The resulting far field sound pressure becomes:

$$p_F(R) = p_D(R) + \sum_{n=1}^{8} p_n(R)$$

Inserting the expressions:

$$p_F(R) = \frac{i\omega Q}{2\pi R}\exp(-ikR) - 0.3950\frac{i\omega Q}{2\pi R}\exp(-1.19ikr)$$

Hence, the far-field sound pressure for $kr < 10$:

$$\boxed{p_F(R) = \left[1 - 0.395\sum_{k=1}^{N}\exp(-1.19ikr)\right] p_D(R)}$$

This is almost the same equation as for the circular baffle; the amplitude of the reflections is reduced from 0.500 to 0.395 and the phase due to the delay is 19 % faster. This could indicate that the model is over-simplified.

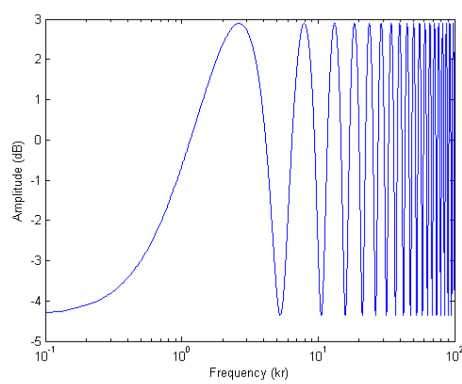


Figure 58 – Diffraction from a square baffle with side length 2*r*. The model is valid for *kr* < 10.

The peak-to-peak ripple is reduced from 9.5 dB (3.5 dB to –6.0 dB) for the circular baffle to 7.3 dB (2.9 dB to –4.4 dB) with the square baffle.

3.5. Listening angle

A loudspeaker system with a crossover network outputs signals from two loudspeakers, which are displaced vertically or horizontally. Listening at angles different from 0° introduces a time delay between the loudspeakers causing interference between the loudspeakers. The problem can be solved by arranging the loudspeakers in-line on the front plane but the problem remains for the other axis and will be analysed below.

3.5.1. Two loudspeakers

Two loudspeakers are placed at the same baffle with the distance L between centres. They are assumed ideal transducers in the following to simplify the analysis.

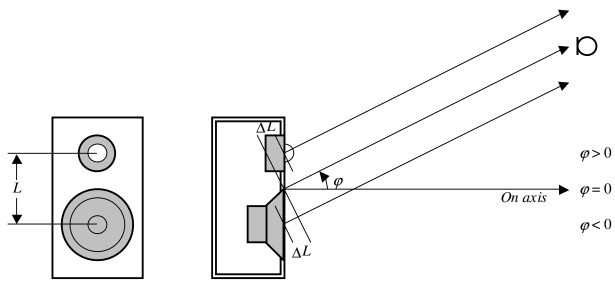


Figure 59 – Loudspeakers are located above each other to avoid horizontal time delays. Vertical offset of the listening angle introduces a time delay thus changing the phase difference between the loudspeakers at the summing position.

Offset angle φ introduces a distance ΔL, which increases the listening distance for one of the loudspeakers and reduces the distance for the other.

$$\Delta L = \frac{L}{2}\sin(\varphi)$$

For L = 0.25 m between the loudspeakers and φ = 15° offset the distance is ΔL = 32 mm.

The distance gives rise to a time shift, which is positive for the channel where the distance is increased and negative for the other.

$$\tau_\varphi = \frac{\Delta L}{c} = \frac{L}{2c}\sin(\varphi)$$

For L = 0.25 m and φ = 15° the time delay is τ_φ = 95 µs.

Time shifting is equivalent to phase.

$$\theta = \omega\tau_\varphi = \frac{\omega L}{2c}\sin(\varphi) = \frac{kL}{2}\sin(\varphi)$$

With angle φ positive upward, and with top position T and bottom position B, the phase relations for the two channels become:

$$\boxed{\begin{aligned} H_T &= \exp(i\theta) = \exp\left(\tfrac{1}{2}ikL\sin(\varphi)\right) \\ H_B &= \exp(-i\theta) = \exp\left(-\tfrac{1}{2}ikL\sin(\varphi)\right) \end{aligned}}$$

This is significant for most of the frequency range since $kL < 0.1$ is only satisfied for frequencies below 20 Hz (L = 0.25 m) and listening angles above ±5° must be expected.

The result is shown in Figure 60 for a first-order crossover with ±15° listening angle corresponding to ±95 µs of time delay (0.25 m loudspeaker distance). In Figure 61 and Figure 62 are crossovers of second and third order with ±15° and ±30° listening angle. The figures apply to positive and negative angles as well.

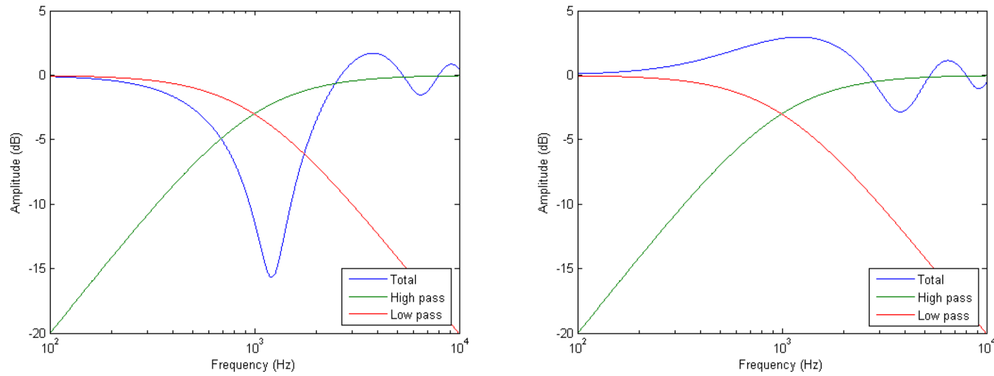


Figure 60 – Amplitude response with angle 15° (left) and –15° (right) for first-order crossover.

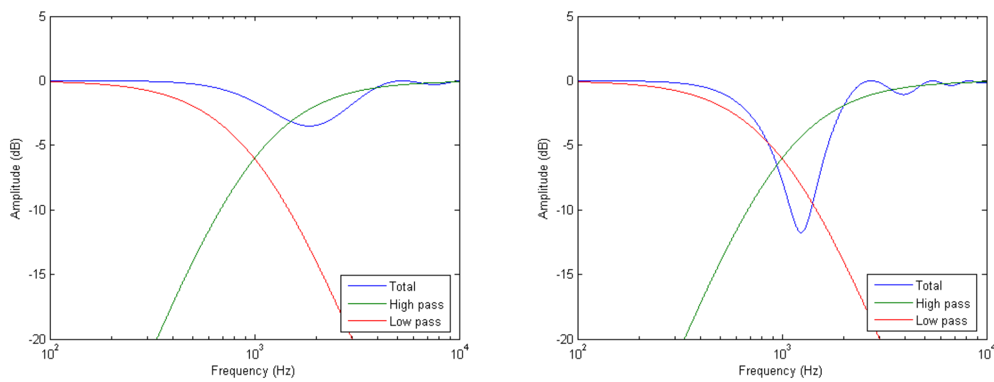


Figure 61 – Amplitude response with angle ±15° (left) and ±30° (right) for second-order crossover with inverted treble.

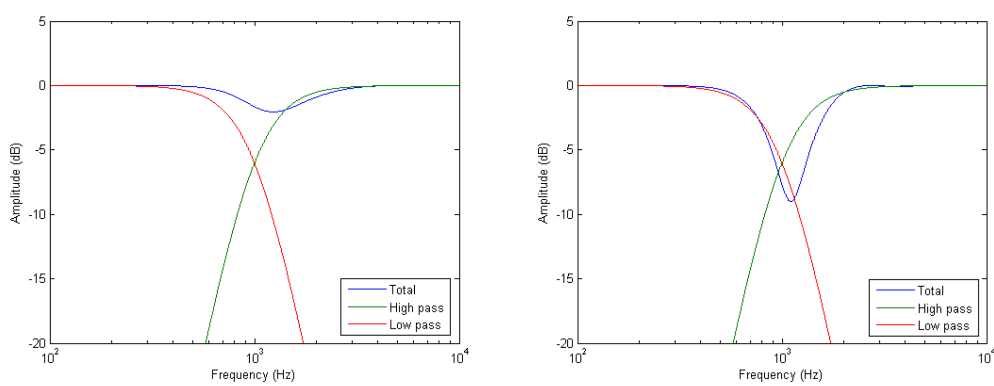


Figure 62 - Amplitude response with angle ±15° (left) and ±30° (right) for fourth-order crossover.

The impact upon amplitude response is significant so the loudspeaker should be tilted to point toward the listener. This can be problematic for public-address applications, where the loudspeaker must cover a large area.

The calculations are equally valid for loudspeaker displacement offset, where the loudspeakers are axially displaced. This is shown to the right of Figure 62.

3.5.2. Three loudspeakers

The phase difference between the bass and treble loudspeakers can be cancelled by using two bass loudspeakers centred around the treble loudspeaker.

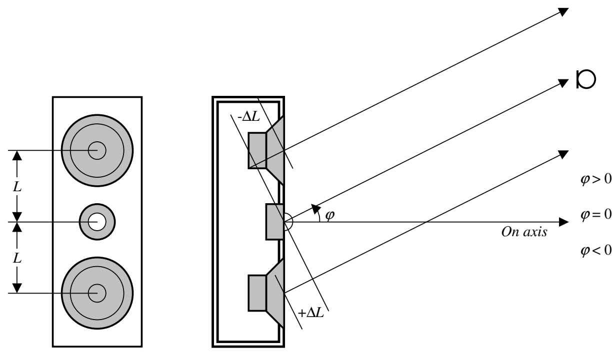


Figure 63 – Two bass loudspeakers balance the time delay to zero at low frequencies.

Offset angle φ introduces a distance ΔL, which increases the listening distance for one of the bass loudspeakers and reduces the distance for the other. The distance between the bass loudspeakers is $2L$ so the equation for the distance becomes.

$$\Delta L = L\sin(\varphi)$$

Hence the phase:

$$\theta = \frac{\omega L}{c}\sin(\varphi) = kL\sin(\varphi)$$

The transfer function for the output from the combined bass loudspeakers is:

$$\boxed{H_B = \frac{1}{2}\exp(i\theta) + \frac{1}{2}\exp(-i\theta) = \cos(kL\sin(\varphi))}$$

So, the phase angle is transformed into an amplitude error, which is zero for on-axis listening and reduces the bass loudspeaker amplitude around crossover. The amplitude error is a function of frequency, and oscillate for high frequencies. The first null occur at the frequency where $kL\sin(\varphi)$ becomes $\pi/2$, which is:

$$f_N = \frac{c}{4L\sin(\varphi)}$$

For L = 0.25 m is the first null at f_N = 1.3 Hz at φ=.15° and f_N = 690 Hz at φ=.30°.

The result is shown in Figure 64 for the first-order network, which now accepts much larger listening angle. The reduction in bass loudspeaker amplitude is clearly seen and the ripples in the high-frequency range is due to interference between the bass loudspeakers.

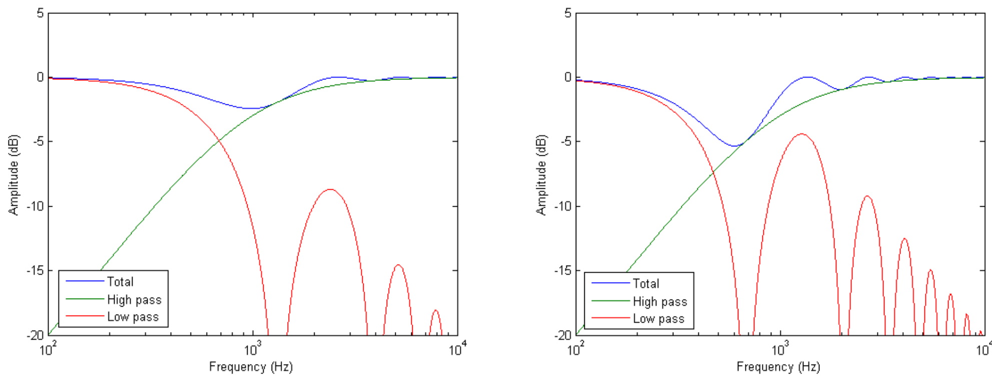


Figure 64 – Amplitude response with angle ±15° (left) and ±30° (right) for first-order crossover.

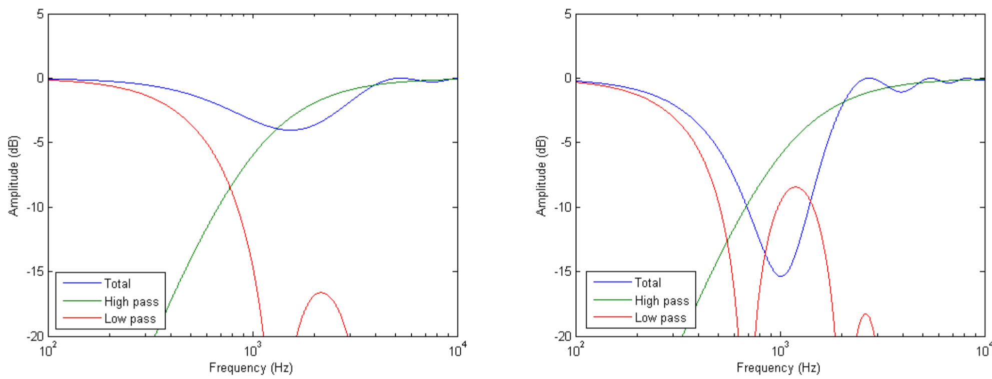


Figure 65 – Amplitude response with angle ±15° (left) and ±30° (right) for second-order crossover with inverted treble.

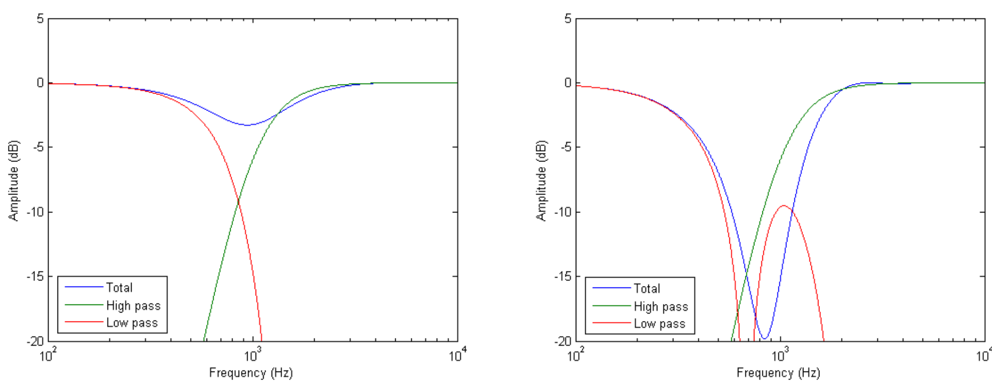


Figure 66 - Amplitude response with angle ±15° (left) and ±30° (right) for fourth-order crossover.

3.6. Boundary reflection

Loudspeakers used within rooms are affected by reflections from the surfaces, which interferes with the direct signal causing peaks and dips in the amplitude response. It is common to use the ray method for the analysis of this problem, see Figure 67. The sound ray from the loudspeaker is reflected by the surface and echoed back into the room as if the surface were a mirror. The resultant sound pressure is calculated by addition of the individual sound rays from the loudspeaker and the virtual sources. The ray method assumes that the location of source and observation point are in the free field, thus ignoring near-field effects such as the increase in radiation impedance of two coherent sources located close to each other. This limits the validity of the ray method to frequencies where the distance between sources, surfaces and observation point must all be large compared to wavelength.

3.6.1. One reflecting surface

A simple model will introduce the ray method before a more complete model is developed. Only one boundary is present, typically the floor or one of the walls. The loudspeaker is assumed pointing directly at the listener at horizontal distance L from the loudspeaker. The listener is distance H_0 from the boundary and the loudspeaker is distance H_1 from the boundary. The boundary is assumed rigid so that all signal is reflected and the loudspeaker directivity is described as defined previously by angle θ.

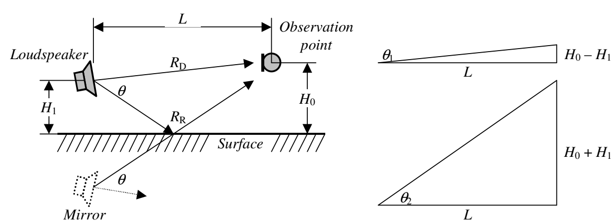


Figure 67 – A simplified model with one reflecting boundary. Direct signal path length is R_D and reflected signal path length is R_R. The loudspeaker is assumed pointing to the listener.

The direct sound at the observation point, located at distance R_D from a monopole source with volume velocity Q, is:

$$p_D = \frac{i\omega\rho Q}{4\pi R_D}\exp(-ikR_D), \quad where \quad R_D = \sqrt{(H_0 - H_1)^2 + L^2}$$

The reflection is delayed due to the increased path length and becomes:

$$p_R = \frac{i\omega\rho Q}{4\pi R_R}\exp(-ikR_R)D(\theta), \quad where \quad R_R = \sqrt{(H_0 + H_1)^2 + L^2}$$

The loudspeaker radiation toward the boundary is dependent upon the angle θ, which can be described as the sum of two angles according to the drawings at the right side of Figure 67:

$$\theta = \theta_1 + \theta_2 = \arctan\left(\frac{H_0 - H_1}{L}\right) + \arctan\left(\frac{H_0 + H_1}{L}\right)$$

The resultant sound pressure becomes:

$$p_D = \frac{i\omega\rho Q}{4\pi R_D}\exp(-ikR_D) + \frac{i\omega\rho Q}{4\pi R_R}\exp(-ikR_R)D(\theta)$$

This can be written:

$$p_R = \left[1 + \frac{R_D D(\theta)}{R_R}\exp(-ik(R_R - R_D))\right]p_D$$

The effect of boundary reflection is to multiply the direct sound pressure by a complex phasor with an amplitude between 0 and 2 and a phase, which is a function of frequency and distances. At low frequencies is the exponential function approaching unity, so the amplitude is increased, with a maximum of 2 times (6 dB) the direct signal. At higher frequencies will the amplitude ripple between a low value where the exponential is –1 and a high value where the exponential is 1.

The first null occur at:

$$k(R_R - R_D) = \frac{\omega(R_R - R_D)}{c} = \pi \quad \Rightarrow \quad f_{NULL} = \frac{c}{2(R_R - R_D)}$$

For R_D = 2 m and R_R = 2.8 m the first null is at f_{NULL} = 210 Hz. The first peak occur at two times f_{NULL}, which is at 420 Hz.

The result is shown in Figure 68 for two different set-ups. The left hand figure is with both loudspeaker and listener at 1 m height and 2 m between loudspeaker and listener. The reflected signal is delayed 0.83 m and the first null occurs at 210 Hz. The ripple amplitude decays at high frequencies since the loudspeaker becomes directive. The right hand figure is with the loudspeaker at 0.2 m distance from the floor. The reflected signal is now only delayed 0.17 m so the first null occurs at 960 Hz and loudspeaker directivity removes most of the ripple.

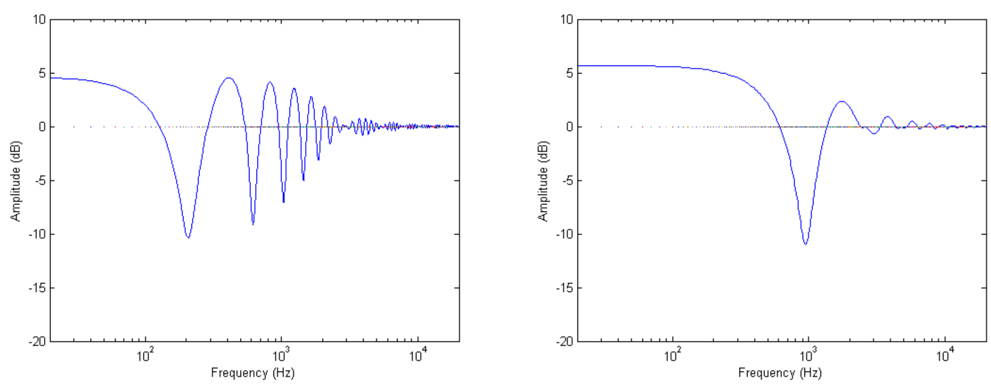


Figure 68 – Amplitude response with loudspeaker at 1 m height (left) and 0.2 m (right), with 2 m listening distance and 1 m listener height. Loudspeaker radius was 0.1 m.

3.6.2. Rectangular room

A more involved model will be developed, which included up to six reflecting boundaries describing a conventional rectangular room.

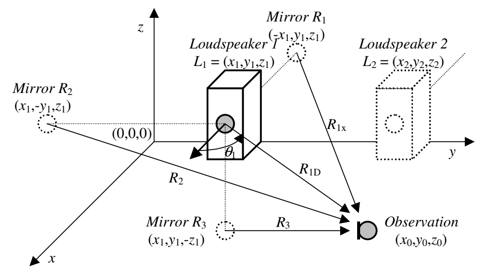


Figure 69 – Loudspeaker 1 is at location $L_1 = (x_1,y_1,z_1)$, direction is along the x-axis and the observation point is at location $L_0 = (x_0,y_0,z_0)$. Mirror source R_1 is located behind the loudspeaker at $R_1 = (-x_1,y_1,z_1)$ since the x-axis is normal to the yz-surface, and so forth for the other reflections R_2 and R_3. Reflection boundaries at (x_L,y_L,z_L) introduce a mirror source at $R_4 = (x_L + x_1,y_1,z_1)$ and so forth for reflections R_5 and R_6.

Reflecting boundaries are assumed located as the sides of a rectangular box with one corner at (0, 0, 0) and another at (x_L, y_L, z_L), thus defining the size of the room. The six surfaces will be referenced through a boundary number b = 1 to 6, as listed below.

Boundary, b	Surface	Comment
1	YZ-plane with $x = 0$	Wall behind loudspeaker
2	XZ-plane with $y= 0$	Left wall
3	XY-plane with $z = 0$	Floor
4	YZ-plane with $x = x_L$	Wall behind listener
5	XZ-plane with $y= y_L$	Right wall
6	XY-plane with $z = z_L$	Roof

With the loudspeaker and observation point located as shown and assuming that the coordinates (x_n, y_n, z_n) are all within the room ($0 < x_n < x_L$, and so forth), the path P_{nD} from loudspeaker n to the listener is defined as the distance from the source location vector L_n to the observation point location vector L_0:

$$P_{nD} = L_0 - L_n = \begin{pmatrix} x_0 - x_n & y_0 - y_n & z_0 - z_n \end{pmatrix}$$

Index n is the loudspeaker number.

Reflected signal paths are represented as P_{nRb} for a path from loudspeaker n reflected through boundary b to the listener. The reflection path vectors for reflection in surfaces with one corner at (0, 0, 0) are:

$$P_{nR1} = \begin{pmatrix} x_0 + x_n & y_0 - y_n & z_0 - z_n \end{pmatrix}$$
$$P_{nR2} = \begin{pmatrix} x_0 - x_n & y_0 + y_n & z_0 - z_n \end{pmatrix}$$
$$P_{nR3} = \begin{pmatrix} x_0 - x_n & y_0 - y_n & z_0 + z_n \end{pmatrix}$$

And for reflection in surfaces with one corner at (x_L, y_L, z_L):

$$\begin{aligned}
P_{nR4} &= \begin{pmatrix} x_0 - (2x_L - x_n) & y_0 - y_n & z_0 - z_n \end{pmatrix} \\
P_{nR5} &= \begin{pmatrix} x_0 - x_n & y_0 - (2y_L - y_n) & z_0 - z_n \end{pmatrix} \\
P_{nR6} &= \begin{pmatrix} x_0 - x_n & y_0 - y_n & z_0 - (2z_L - z_n) \end{pmatrix}
\end{aligned}$$

Loudspeaker radiation is dependent upon the angle from on-axis so the direction of the loudspeaker must be specified. Loudspeaker n is orientated with the main direction (loudspeaker front side) pointing along the direction vector L_{nD}:

$$L_{nD} = \begin{pmatrix} x_{nD} & y_{nD} & z_{nD} \end{pmatrix}$$

The coordinates can be determined from the horizontal angle θ, which is 0° along the x-axis and the vertical angle ϕ, which is 0° at the xy-plane ($z = 0$), as:

$$\begin{aligned}
x_{nD} &= \cos(\theta_n)\cos(\phi_n) \\
y_{nD} &= \sin(\theta_n)\cos(\phi_n) \\
z_{nD} &= \sin(\phi_n)
\end{aligned}$$

For loudspeaker 1 pointing along the x-axis the vector becomes L_{1D} =(1, 0, 0).

The observation angle is different for the reflections; one coordinate changes sign due to the reflection.

$$\begin{aligned}
L_{nR1} = L_{nR4} &= \begin{pmatrix} -x_{nD} & y_{nD} & z_{nD} \end{pmatrix} \\
L_{nR2} = L_{nR5} &= \begin{pmatrix} x_{nD} & -y_{nD} & z_{nD} \end{pmatrix} \\
L_{nR3} = L_{nR6} &= \begin{pmatrix} x_{nD} & y_{nD} & -z_{nD} \end{pmatrix}
\end{aligned}$$

The observation angle for the direct sound from the loudspeaker θ_{nD} is determined as the angle between the loudspeaker direction vector L_{nD} and the vector pointing from the loudspeaker to the observation point P_{nD}. The angle is 0° for the loudspeaker pointing directly at the observation point. The observation angle is extracted from the definition of the inner product (dot product) between L_{nD} and P_{nD} [5]:

$$L_{nD} \bullet P_{nD} = |L_{nD}||P_{nD}|\cos(\theta_{nD})$$

The observation angle for the direct signal from loudspeaker n becomes:

$$\theta_{nD} = \arccos\left(\frac{L_{nD} \bullet P_{nN}}{|L_{nD}||P_{nD}|}\right)$$

The inner product is defined in MATLAB with A • B expressed as A’ * B, where A’ means the transposed of column vector A.

The observation angles for the reflections are:

$$\theta_{nR1} = \arccos\left(\frac{L_{nR1} \bullet P_{nR1}}{|L_{nR1}||P_{nR1}|}\right) \quad \theta_{nR2} = \arccos\left(\frac{L_{nR2} \bullet P_{nR2}}{|L_{nR2}||P_{nR2}|}\right) \quad \theta_{nR3} = \arccos\left(\frac{L_{nR3} \bullet P_{nR3}}{|L_{nR3}||P_{nR3}|}\right)$$

$$\theta_{nR4} = \arccos\left(\frac{L_{nR4} \bullet P_{nR4}}{|L_{R4n}||P_{nR4}|}\right) \quad \theta_{nR5} = \arccos\left(\frac{L_{nR5} \bullet P_{nR5}}{|L_{nR5}||P_{nR5}|}\right) \quad \theta_{nR6} = \arccos\left(\frac{L_{nR6} \bullet P_{nR6}}{|L_{nR6}||P_{nR6}|}\right)$$

The sound pressure at the observation point from loudspeaker *n* and six reflections, assuming equal volume velocity *Q* for all loudspeakers and mirror sources, is:

$$p_n = \frac{i\omega\rho Q}{4\pi P_{nD}} \exp(-ikP_{nD}) D(\theta_{nD}) + \sum_{b=1}^{6} \frac{i\omega\rho Q}{4\pi P_{nRb}} \exp(-ikP_{nRb}) C_{Rb} D(\theta_{nRb})$$

The reflection coefficient C_R was included for two reasons; it is an easy way to selective enable and disable reflections and it allows use of partially reflective surfaces. Set all coefficients to zero to completely cancel reflection or set one or more to unity to selectively enable reflection. Values between zero and unity simulates panel absorbers, thick curtains or walls with large openings and the constants may be complex, if required. $D(\theta)$ represents the loudspeaker directivity, which is defined in 3.3.

The resultant sound pressure from *N* loudspeakers with six reflecting surfaces:

$$p = \frac{i\omega\rho Q}{4\pi} \sum_{n=1}^{N} \left[\frac{D(\theta_n)}{P_n} \exp(-ikP_n) + \sum_{b=1}^{6} \frac{C_b D(\theta_{nRb})}{P_{nRb}} \exp(-ikP_{nRb}) \right]$$

3.6.3. Home entertainment

A typical set-up for home entertainment could be with the loudspeaker 1 m above ground and the listener at 2 m distance and with the head at the same height as the loudspeaker.

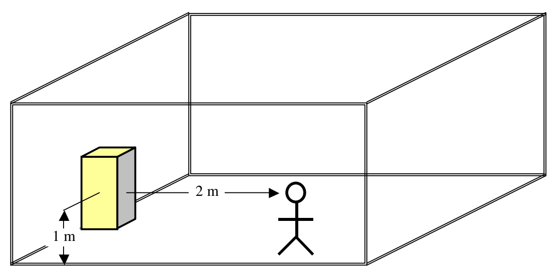


Figure 70 – Set-up for home entertainment with a loudspeaker in a living room, which was 5 m × 10 m and 3 m high in the analysis.

The amplitude response is shown in Figure 71, left. The reflection is delayed 0.82 m corresponding to a half wave length at 210 Hz where destructive interference occurs. Lower frequencies are increased due to the reflection being in-phase and higher frequencies oscillates. An improvement for the low-frequency operation can be obtained by reducing the loudspeaker elevation as shown to the right.

The curves are similar to Figure 68, which was calculated using the simplified model with one reflecting boundary. The difference in Figure 71, right, is due to the loudspeaker pointing along the *x*-axis and not directly toward the listener.

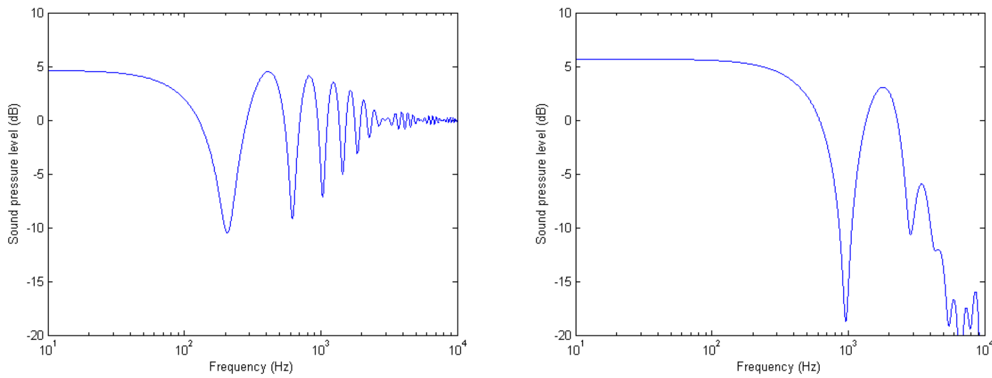


Figure 71 – Amplitude response for loudspeaker at 1 m height and 2 m distance (left) and height reduced to 0.2 m (right). Listening height was 1 m. Diaphragm radius was *a* = 0.1 m and the loudspeaker points along the *x*-axis.

The oscillations ceases at high frequencies due to the frequency-dependent loudspeaker directivity and the small bursts of oscillation at high frequency is caused by side lobes. Anyway, the model is not valid for high frequencies where the diaphragm is not vibrating as a rigid piston, the limit is around $ka < 3$, which is 1.6 kHz for a diaphragm diameter of $a = 0.1$ m.

Activating reflection from the side wall with 1 m distance from the loudspeaker and listener reintroduces the reflection with the dip at 210 Hz but the dip is less pronounced now since the overall level is increased. Moving the loudspeaker closer to the wall increases the frequency of the dip.

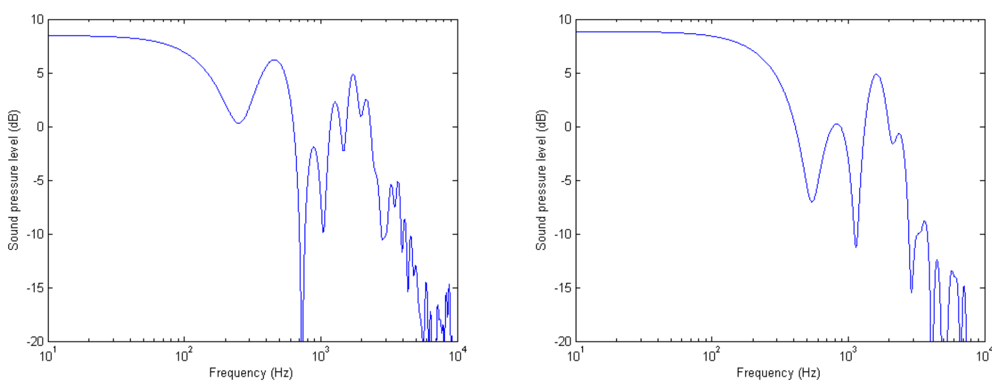


Figure 72 – Amplitude response for loudspeaker at 0.2 m height, 1 m from a wall and at 2 m listener (left). Distance to wall reduced to 0.5m (right).

The high-frequency response is more ragged than before since three signal paths are combined (direct sound and two reflections). Activating the reflection from the wall behind the loudspeaker increases the low-frequency level.

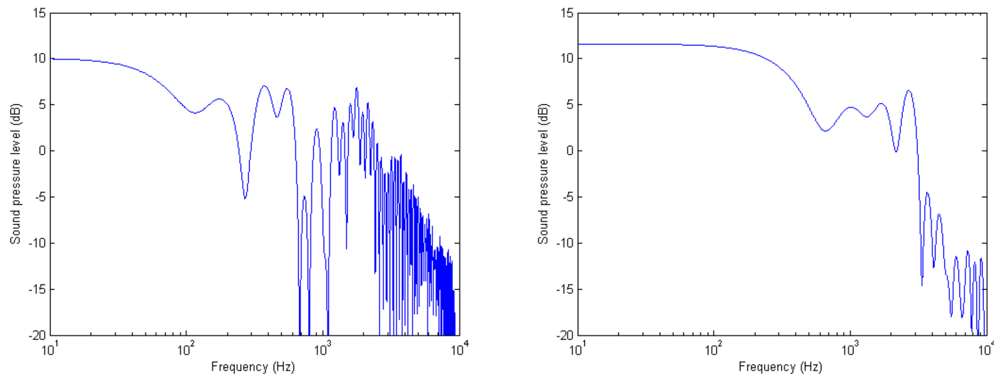


Figure 73 – Amplitude response for loudspeaker at 0.2 m height, 1 m from both walls and at 2 m to listener (left). Distance to all three surfaces reduced to 0.2 m (right).

A descent amplitude response is obtained with the loudspeaker located close to both the corner between floor and walls as shown in Figure 73, right. The amplitude response indicates the need of crossover to the midrange loudspeaker below 200 Hz so the corner is optimal for a sub woofer (ignoring the standing waves within a real room).

The midrange and treble loudspeakers should be displaced from the sub woofer in order to reduce the effect of room reflections.

All surfaces are activated in Figure 74 where the loudspeaker is located at some distance from the corner at the left drawing and is moved closer to the corner in the second drawing. The most flat amplitude response is obtained close to the corner, but using the corner position may cause excitation of room resonances, not modelled here, and this may lead to a booming reproduction of low frequencies.

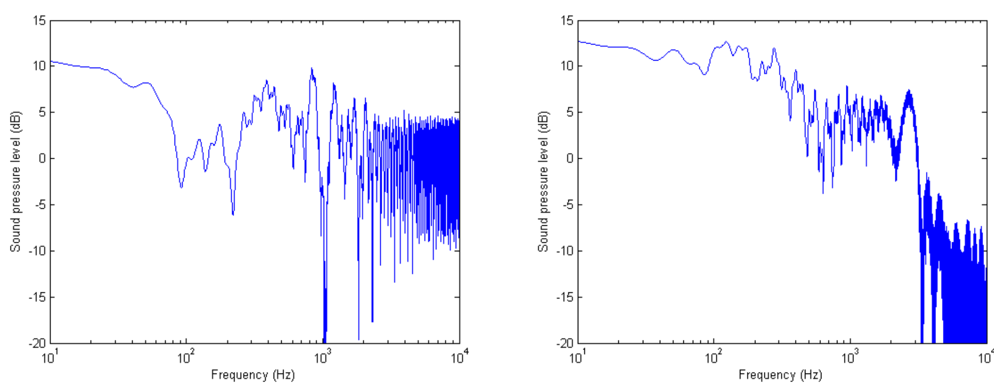


Figure 74 – Amplitude response for a room of 5 m by 10 m with 3 m height. Loudspeaker at 1 m distance from corner (left) or 0.2 m from corner (right).

It is interesting to note, that the level at low frequencies is not increased 3 dB for each surface, as is sometimes stated in the popular literature; the figures are rather 6 dB from the first surface, 3 dB from the two next, and an insignificant amount from additional surfaces. There is no simple explanation to this observation – it could be expected that the direct and reflected signals would add almost lossless at low frequencies, but it appears not to be so.

3.6.4. Public address

Another application is a large room for theatre or concerts, which is fitted with a loudspeaker system for public address.

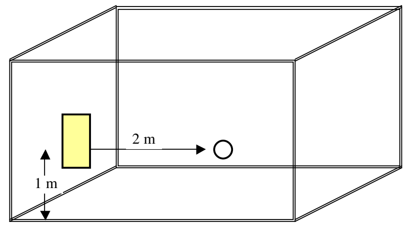


Figure 75 – Large room for concerts, with dimensions 50 m × 30 m and 15 m high. The loudspeaker is located 5 m from each wall and 10 m above the floor and the listening position was 35 m from the loudspeaker, 1/3 from the wall and 3 m above the floor. The surfaces were absorbing.

The resulting amplitude response for a room with reflecting surfaces is shown in Figure 76 for a room without damping material. Only the direct signal and six reflections are included but the amplitude response is very ragged with ±10 dB variation within the main frequency range. The effect of adding damping to the surfaces is shown to the right, where the amplitude response is within ±5 dB for most of the audible range but the overall level is reduced some 10 dB since the reflections are reduced in level.

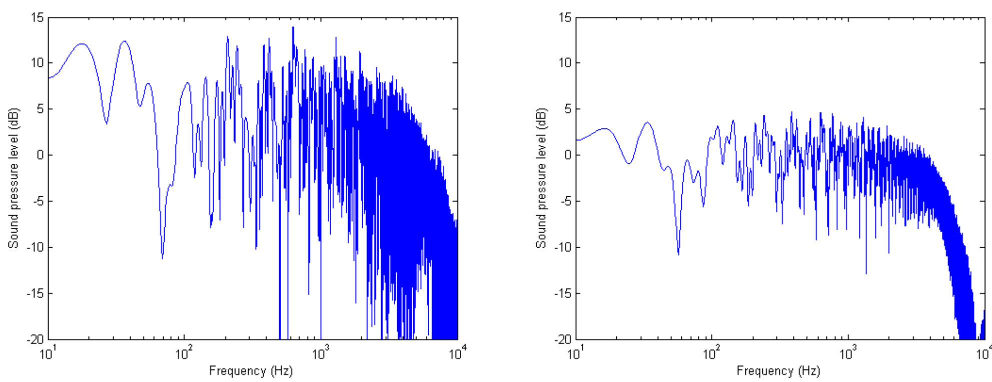


Figure 76 – Amplitude response for a room without damping (left) and the reflection coefficients adjusted to 0.3 for the wall behind loudspeaker and the left and right walls and 0.1 for roof, floor and the wall behind the listener (right).

The decaying high-frequency amplitude is caused by the directivity of the loudspeaker; the listener is off-axis.

3.7. Loudspeaker characteristics

Four loudspeakers have been selected as being representative for the loudspeaker sizes found in two-way and three-way systems and are listed in Table 3 together with the published parameters of interest for this study. The values are not directly used in this document but have formed the basis for selection of parameters.

Table 3 – Typical loudspeaker characteristics for two-way and three-way systems.

Loudspeaker model P:Peerless, S:ScanSpeak	**Symbol**	**Woofer** S:26W8667	**Bass** P:205WR33	**Midrange** S:13M8636	**Treble** S:DT2905	**Unit**
Diaphragm diameter	D		168			mm
Effective diaphragm area	S_D	$32\ 10^{-3}$	$22\ 10^{-3}$	$4.8\ 10^{-3}$	$0.85\ 10^{-3}$	m^2
Voice coil resistance	R_E	5.8	6.0	5.8	4.7	Ω
Voice coil inductance	L_E	0.4	1.8	0.1	0.08	mH
Force factor	BL	11.1	9.6	6.0	3.5	N/A
Diaphragm & voice coil mass	M_{MD}	56	25.9	4.6	0.45	g
Suspension compliance	C_{MS}	-	1.13	-	-	mm/N
Suspension friction loss	R_{MS}	1.5	-	0.8	-	kg/s
Sensitivity (2.83 V)	L_{ref}	87	90	86.5	90	dB
Linear excursion	x_{MAX}	±9.0	±5.5	±1.5	±0.4	mm
Electrical quality factor	Q_{MS}	5.2	3.94	2.8	-	-
Mechanical quality factor	Q_{ES}	0.36	0.31	0.36	-	-
Total quality factor	Q_{TS}	0.34	0.29	0.32	-	-
Equivalent volume	V_{AS}	136	76	3.0	-	m^3
Resonance frequency	f_S	22	30	77	650	Hz
Frequency for $ka = 1$	f_A	0.54		1.4	3.3	kHz
Frequency due to voice coil	f_C	2.3		9.2	9.4	kHz

The top values refers to the published figures for the stated loudspeakers. The bottom two values are calculated using the equations from this section.

3.8. Group delay

Group delay specify the delay experienced by a group of sinusoidal components, which have frequencies within a narrow frequency interval about $f = \omega/2\pi$. The bandwidth in this interpretation must be restricted to a frequency interval over which the phase response is approximately linear.

3.8.1. Calculation method

The group delay is defined as the rate of change of phase with respect to frequency [7]:

$$\tau_{GD} = -\frac{d\theta}{d\omega}$$

The phase is a property of the transfer function $H(\omega)$, which can be written in polar notation with $G(\omega)$ as the amplitude response and $\theta(\omega)$ as the phase response where both are real valued functions:

$$H(\omega) = G(\omega)\exp(i\theta(\omega))$$

This can be separated into amplitude and phase terms using the logarithmic function.

$$\ln(H(\omega)) = \ln(G(\omega)) + i\theta(\omega)$$

This is differentiated with respect to angular frequency:

$$\frac{d}{d\omega}\ln(H(\omega)) = \frac{d}{d\omega}\ln(G(\omega)) + i\frac{d}{d\omega}\theta(\omega)$$

Using the differentiation rule for the logarithm (*Schaum* 13.27):

$$\frac{H'(\omega)}{H(\omega)} = \frac{G'(\omega)}{G(\omega)} + i\theta'(\omega)$$

$G'(\omega)$ and $\theta'(\omega)$ denotes the derivative of $G(\omega)$ and $\theta(\omega)$ respectively. The group delay is represented by the imaginary part, and since $G'(\omega)/G(\omega)$ is real this becomes.

$$\boxed{\tau_{GD} = -\theta'(\omega) = -\mathrm{Im}\left\{\frac{H'(\omega)}{H(\omega)}\right\}}$$

3.8.2. Implementation in MATLAB

The derivative of $H(\omega)$ can be expressed as the slope of $H(\omega)$ within a narrow frequency range from ω to $\omega + \Delta\omega$ by use of the definition of the derivative (*Schaum* 13.1):

$$H'(\omega) = \lim_{\Delta\omega \to 0} \frac{H(\omega + \Delta\omega) - H(\omega)}{\Delta\omega}$$

The difference $\Delta\omega$ cannot reach zero so a finite value of 0.001 will be used.

> The frequency variable is $s_0 = i\omega/\omega_0 = if/f_0$ so the increment becomes $\Delta\omega = 0.001\omega_0$ or alternatively $\Delta f = 0.001f_0$. The incremental frequency is thus 1 Hz for a normalisation frequency (crossover frequency) of $f_0 = 1000$ Hz.

Hence, calculation of $H(\omega)$ using the following expression:

$$\boxed{\tau_{GD} = -\theta'(\omega) = -\operatorname{Im}\left\{\frac{H(\omega+\Delta\omega) - H(\omega)}{H(\omega)\Delta\omega}\right\}}$$

The frequency is defined as the vector f_0*[0.1:0.001:10], using MATLAB notation, so the incrementally larger frequency uses another vector g defined from f to output the step following immediately after, i.e. defined as $g = f + 0.001 * f_0$.

3.8.3. Verification

Two filters will be analysed for testing the MATLAB implementation of group delay.

The first test object is a first-order low-pass filter, which is brought onto a form suitable for analytic extraction of the phase.

$$H_1 = \frac{1}{1+s_0} = \frac{1}{1+\dfrac{i\omega}{\omega_0}} = \frac{1-\dfrac{i\omega}{\omega_0}}{\left(1+\dfrac{i\omega}{\omega_0}\right)\left(1-\dfrac{i\omega}{\omega_0}\right)} = \frac{1-i\dfrac{\omega}{\omega_0}}{1+\left(\dfrac{\omega}{\omega_0}\right)^2}$$

The phase is:

$$\theta = \arctan\left(\frac{\operatorname{Im}\{H_1\}}{\operatorname{Re}\{H_1\}}\right) = -i\frac{\omega}{\omega_0}$$

And the group delay becomes (*Schaum* 13.22):

$$\tau_{GP} = -\frac{d}{d\omega}\left(-i\frac{\omega}{\omega_0}\right) = \frac{1}{\omega_0}\frac{1}{1+\left(\dfrac{\omega}{\omega_0}\right)^2}$$

The result is shown in Figure 77 and indicates good agreement.

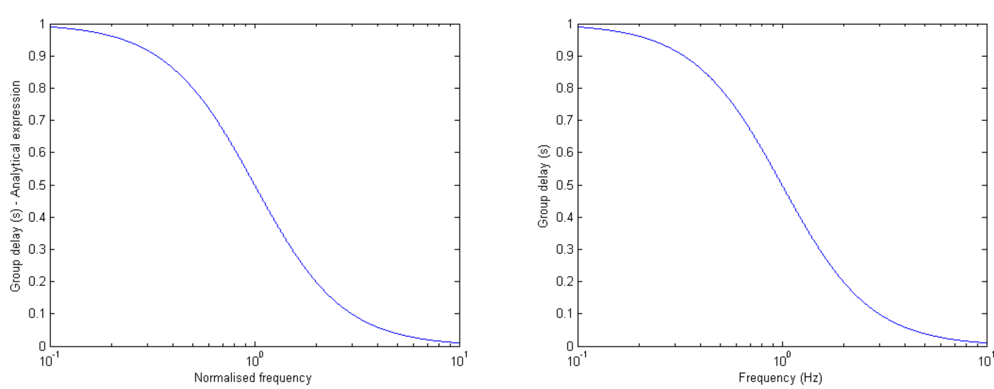


Figure 77 – Group delay using the analytic expression (left) and the calculation from the transfer function of the low-pass filter (right).

The second test object a second-order all-pass filter with constant amplitude for $a_1 = 2$.

$$H_2 = \frac{1 - s_0^2}{1 + a_1 s_0 + s_0^2}$$

The derivative of the test function can be calculated from the definition of the derivative of a quotient of two functions (*Schaum* 13.9):

$$H_2' = \frac{dH_2}{d\omega} = \frac{\left(1 + a_1 s_0 + s_0^2\right)\frac{d}{d\omega}\left(1 - s_0^2\right) - \left(1 - s_0^2\right)\frac{d}{d\omega}\left(1 + a_1 s_0 + s_0^2\right)}{\left(1 + a_1 s_0 + s_0^2\right)^2}$$

The derivative of the frequency variables:

$$\frac{d}{d\omega}\left(s_0\right) = \frac{d}{d\omega}\left(\frac{i\omega}{\omega_0}\right) = \frac{i}{\omega_0}$$

$$\frac{d}{d\omega}\left(s_0^2\right) = \frac{d}{d\omega}\left(\frac{i\omega}{\omega_0}\right)^2 = -\frac{2\omega}{\omega_0^2} = \frac{i2}{\omega_0}\frac{i\omega}{\omega_0} = \frac{i2}{\omega_0} s_0$$

Hence, the derivative of the test function:

$$\boxed{H_2' = \frac{\left(1 + a_1 s_0 + s_0^2\right)\left(-\frac{i2}{\omega_0} s_0\right) - \left(1 - s_0^2\right)\left(\frac{ia_1}{\omega_0} + \frac{i2}{\omega_0} s_0\right)}{\left(1 + a_1 s_0 + s_0^2\right)^2}}$$

This was implemented in MATLAB for comparison and the result of the analytical expression is shown in Figure 78 with the calculation method based on transfer function output shown in Figure 79.

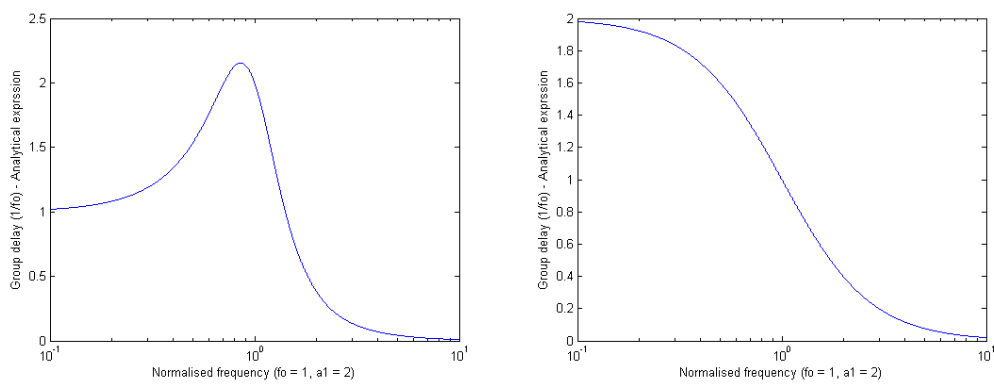


Figure 78 – Group delay using a test function with symbolic differentiation carried out before calculation. Plot for $a_1 = 1$ (left) and $a_1 = 2$ (right).

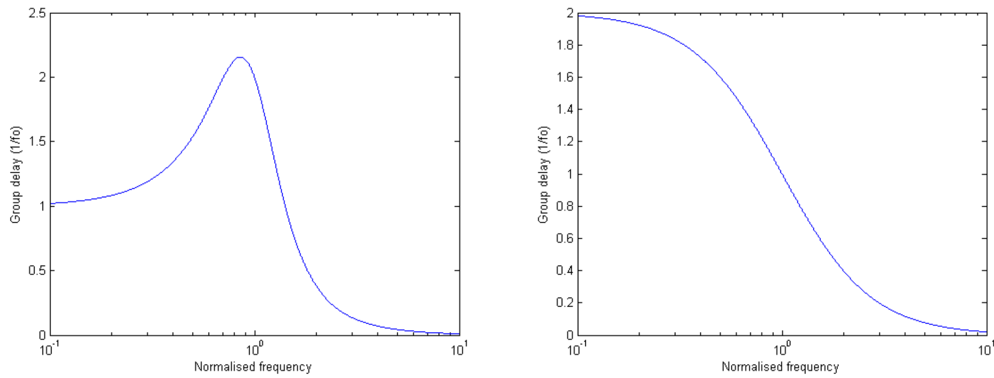


Figure 79 – Group delay using the transfer function output at two frequencies with incremental distance (i.e. f and $f + \Delta f$). Plot for $a_1 = 1$ (left) and $a_1 = 2$ (right).

There are no visual differences between the figures, so the agreement is good. But a couple of figures using some few transfer functions cannot proof the validity of the calculation – but it indicates that the method come close on a couple of tests.

4. Assembling the models

It is time to combine the models into a system design trial. The target is a low-budget two-way system, so the design must be kept simple and uncomplicated to assemble during production, i.e. using a simple passive crossover network.

4.1. Loudspeaker models

The loudspeaker system is a closed cabinet with baffle measures shown in Figure 88. The loudspeaker units are:

- A bass loudspeaker, 8 inch with 70 Hz resonance frequency and a total quality factor of 0.7 within the cabinet and it is assumed to reproduce smoothly to 3 kHz at –3 dB with a cut-off slope of –6 dB/octave at higher frequencies.
- A treble loudspeaker, 1 inch dome with 1 kHz resonance frequency, a total quality factor of 1 and high-frequency roll-off above 20 kHz at – 3 dB.

The amplitude and phase responses of the system are shown in Figure 80 where the "total" curve represents a system where the outputs are added without a crossover filter.

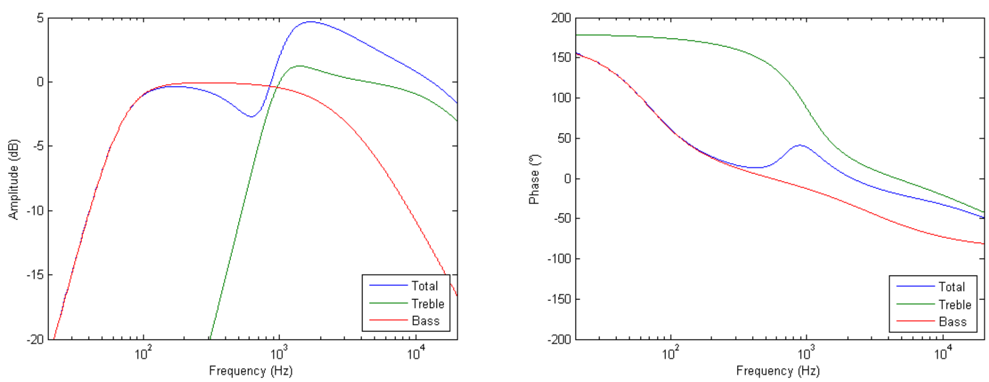


Figure 80 – Loudspeaker models for the two-way system. The bass loudspeaker is defined by the resonance frequency f_S = 70 Hz, total quality factor Q_{TC} = 0.7 and high-frequency roll-off at f_C = 3 kHz. For the treble loudspeaker is f_S = 1 kHz, Q_{TC} = 1.0 and f_C = 20 kHz.

4.2. Crossover network

A crossover network must protect the treble loudspeaker, and this is in this report assumed fulfilled for at least 20 dB of attenuation at the resonance frequency of 1 kHz. Using a second-order crossover network offers 24 dB of attenuation with a crossover frequency of 4 kHz, so this will be used for a start.

A notch at the crossover frequency is to be expected using a second-order crossover network due to 180° of phase difference between the bass and treble loudspeakers so the treble loudspeaker is inverted. Assuming ideal addition of the outputs require a crossover network with 6 dB attenuation at the crossover frequency so the filter must use a quality factor of 0.5 since the attenuation is 20 $\log_{10}(Q)$ dB.

The resulting amplitude and phase responses are shown in Figure 81. A dip of approximately 3 dB is seen at the crossover frequency.

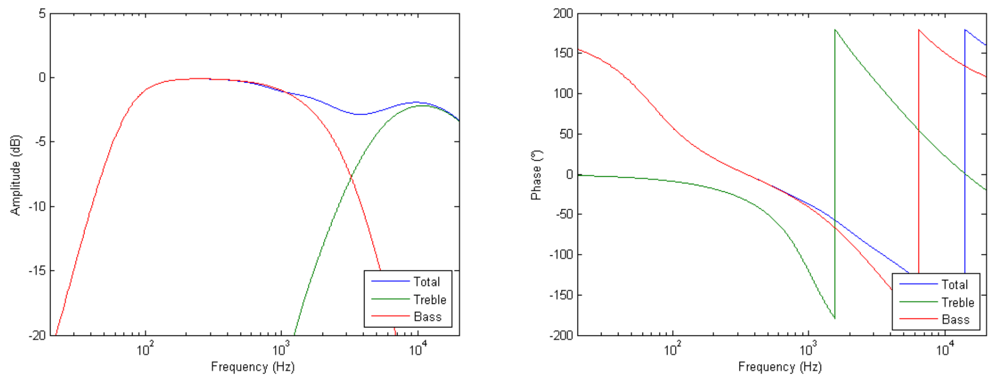


Figure 81 – Resultant amplitude and phase response for the system with a second-order crossover network at 4 kHz and inverted treble.

The phase difference between bass and treble is around 50°, so the loudspeakers are not in-phase due to the phase responses of the loudspeaker units. The difference can be reduced by removing one of the poles from the low-pass filter thus simplifying the crossover network by reducing the low-pass filter to first order. The result is shown in Figure 82, where the phase difference is close to 0° at 4 kHz and the amplitude response is improved.

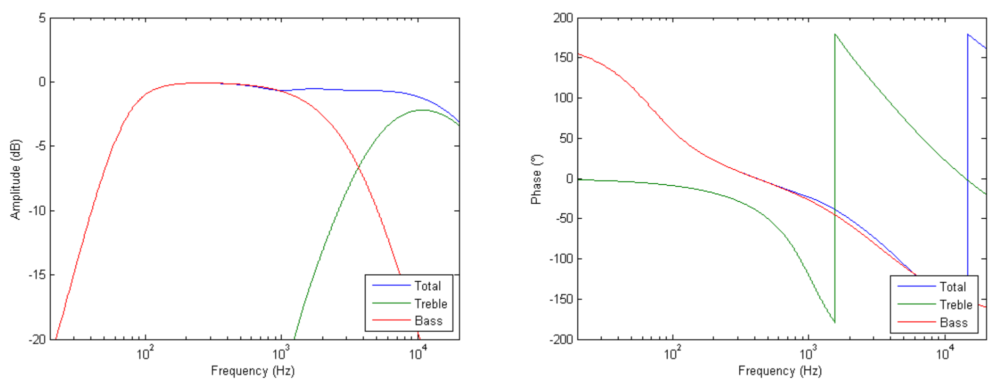


Figure 82 – Loudspeaker response for a system with a first-order low-pass filter for the bass loudspeaker and second-order filter for the treble loudspeaker.

A crossover network with an inductor in series with the loudspeaker is not attractive, first of all due to the loudspeaker impedance, which is increasing at high frequencies and thus opposing the intended low-pass filtering, and second, because the inductor is a relatively costly component.

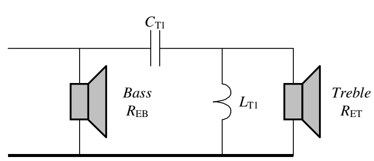


Figure 83 – Passive crossover network for the two-way system. The bass loudspeaker high-frequency roll-off is used as the low-pass channel of the crossover network.

If the inductor is removed, the resulting crossover network consists of a second-order high-pass filter for the treble loudspeaker and no filter for the bass loudspeaker. This is very attractive from a production point of view, but the resulting response in Figure 84 (left picture) shows response peaking some 2 dB around the crossover frequency. This peak will not be removed by the inductor, unless an impedance compensation is included for the bass loudspeaker, and this is not the idea behind the system.

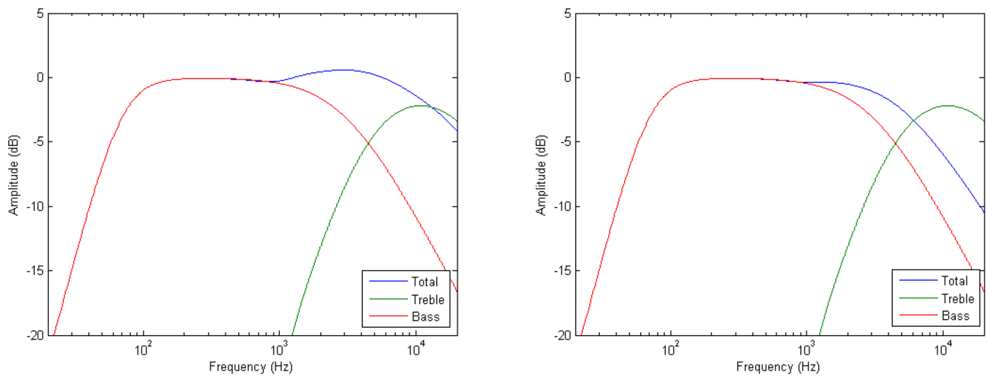


Figure 84 – Loudspeaker response with the low-pass filter removed (left) and with the treble loudspeaker attenuated 6 dB (right). Neither designs are acceptable.

The peak can be removed by attenuation of the treble loudspeaker, but the resultant amplitude response, shown in Figure 84 (right picture), shows that the result is an early roll-off in the treble, which cannot be accepted.

The phase response of the bass channel should include two poles, one from the loudspeaker and another from the crossover network, but the current design only includes the pole from the loudspeaker. The phase is not sufficiently negative, so it could be possible to add the missing phase by time-shifting of the signal from the treble loudspeaker by moving it axially. A time shift introduces a rotating phase where the angular speed is defined by frequency f and the distance ΔL the treble loudspeaker is moved, according to:

$$H_{TIME} = \exp(-i\tau_D f), \quad \tau_D = \frac{\Delta L}{c}$$

A delay of 100 µs corresponds to a distance of ΔL = 34 mm and the result of this movement of the treble loudspeaker is shown to the left in Figure 85. The result is not the intended removal of the peak, rather is the amplitude response completely ruined by the movement. Doing the opposite, advancing time by moving the treble loudspeaker toward the listener, is shown to the right is Figure 85. There is a little change but the peak is not at all removed.

So, it seems that we have to live with the 2 dB boost around 4 kHz.

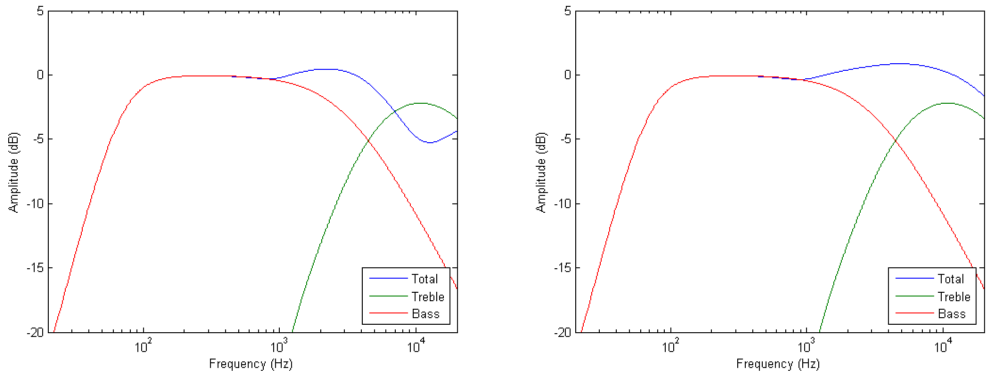


Figure 85 – Loudspeaker response with loudspeaker delayed 100 µs by moving is 34 mm into the cabinet, or the bass loudspeaker 34 mm out from the cabinet (left) and advancing the time 100 µm by moving it the other way (right).

4.3. Angular response

The loudspeaker proved rather sensitive to modest delays, thus indicating that it would be a good idea to study the behaviour to off-axis listening. Two angles are relevant for this analysis, the horizontal angle representing a loudspeaker pointing toward one of the sides of the head of the listener, and the vertical angle representing a loudspeaker pointing above or below the head.

The effect of horizontal angle is shown in Figure 86 for two angles at 15° and 30° and shows that the amplitude response is somewhat sensitive to changes in the horizontal angle. The bass loudspeaker is the problem since the diaphragm diameter is comparable to wavelengths around the crossover frequency.

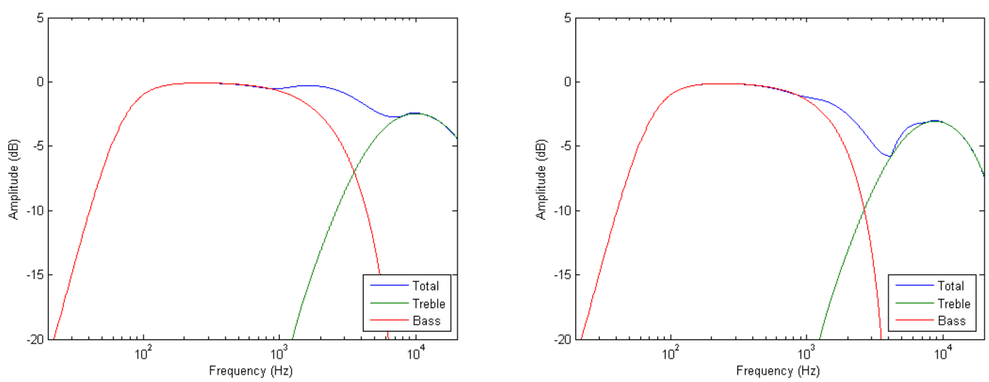


Figure 86 – Loudspeaker response for a horizontal angle of 15° (left) and 30° (right). The loudspeakers radius was 100 mm for the bass loudspeaker and 10 mm for the treble loudspeaker. The bass loudspeaker becomes directive above 550 Hz (*ka* = 1).

The effect of vertical angle is shown in Figure 87 for two angles at 15° and 30° and shows that the amplitude response is quite sensitive to changes in the horizontal angle. The loudspeaker is not symmetrical on the vertical axis, and the behaviour for negative angles is not shown here, but the curves indicate that the loudspeaker must be tilted so that it points to the listener.

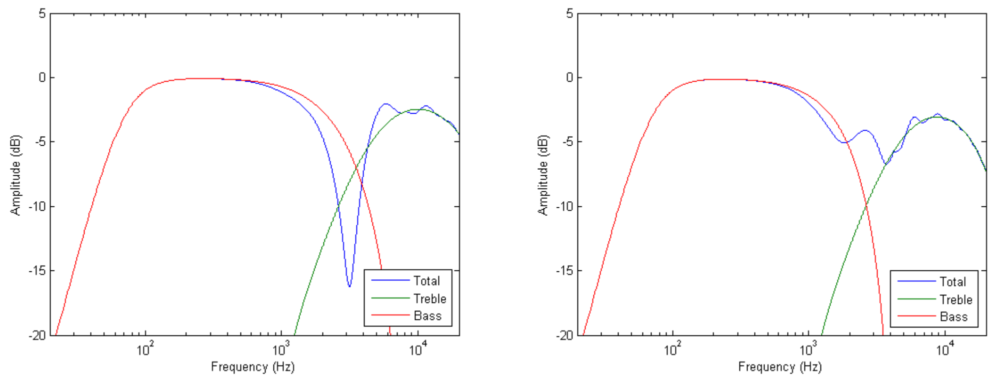


Figure 87 – Loudspeaker response for a vertical angle of 15° (left) and 30° (right).

A conclusion so far is that the peak at the crossover frequency should be kept as it is although the on-axis response could be improved. The consequence of modifying the loudspeaker to a flatter amplitude would most probably be degradation of the off-axis response. The direct sound will be slightly improved around the crossover frequency but the off-axis response shows a decrease in the same range and this affects the sound pressure level within the reverberant field.

4.4. Reflections

Reflection from the edges of the loudspeaker interferes with the loudspeaker and was modelled in section 3.4 where the model with circular sections were most successful and will be used for the analysis below. The loudspeaker cabinet is rectangular and will be modelled by four circular sections with radii equal to the mean distance from the centre of the bass loudspeaker to the edge. This is a coarse model, but it gives an idea of the effect of the front baffle.

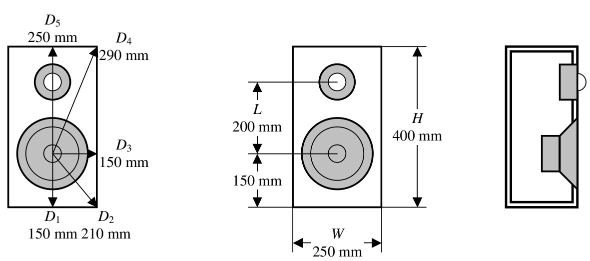


Figure 88 – Loudspeaker front baffle measures for calculation of diffraction.

The shortest mean distance is the average between D_1 and D_2; hence $R_1 = 0.18$ m. Due to symmetry is the two next shortest distances identical and the average between D_2, D_3 and D_4; hence $R_2 = R_3 = 0.22$ m. The largest mean distance is the average between D_4 and D_5; hence $R_1 = 0.27$ m. The result is shown in Figure 89 and is seen to overrule the effect of the peak around the crossover frequency. Also seen is the reduced output at low frequencies where the baffle becomes small compared to wavelength. The loudspeaker is at low frequencies radiating into 4π solid angle, while the baffle limits the radiation angle to 2π at higher frequencies.

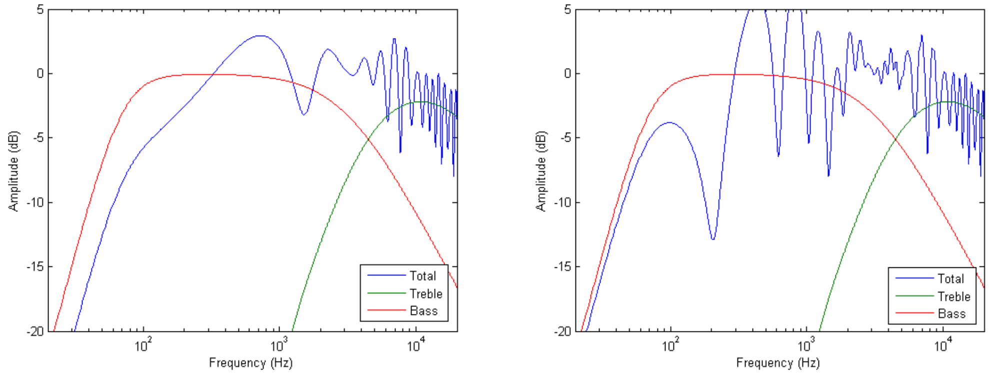


Figure 89 – Loudspeaker response with reflections from the baffle (left) and with reflections from one boundary included (right). The loudspeaker was located 1 m above ground at 2 m distance from the listener.

Reflections from a large surface (the floor) is shown to the right of Figure 89 with the loudspeaker 1 m above ground at 2 m distance to the listener with his or hers ears 1 m above ground. The low frequency response is improved but a dip is seen at 210 Hz, which is due to the delayed distance through the reflection path, which is 0.83 m longer than the direct path and cause destructive interference with the direct signal. Interference becomes constructive at 420 Hz and so forth.

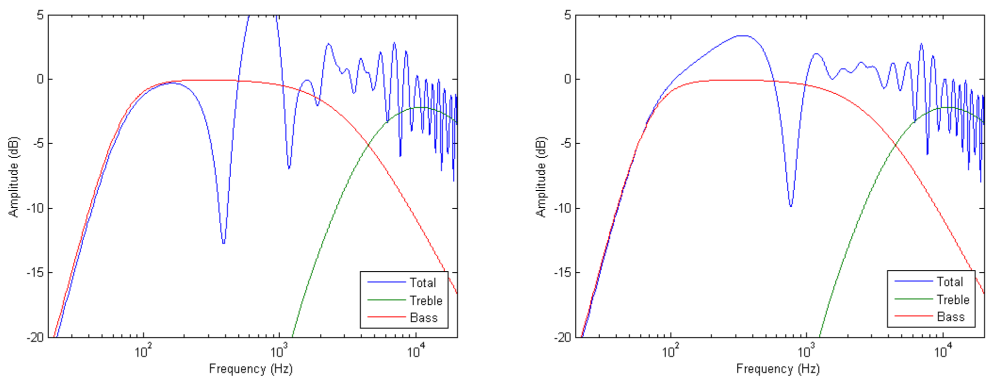


Figure 90 – Loudspeaker at 0.5 m (left) and 0.25 m (right). The model includes the crossover network, the loudspeakers, diffraction and one reflection.

An improvement is obtained by reducing the loudspeaker height to 0.5 m above the floor (Figure 90, left picture). The first dip occur at 350 Hz and the first peak is located at 700 Hz. The peak is of large amplitude but can be reduced by lowering the height to 0.25 m above the floor (Figure 90, right picture).

The high-frequency ripple is due to the simplified models and will be less pronounced in real life.

4.5. Conclusion

The models form an effective tool for initial loudspeaker design although improvements are required especially for the diffraction and boundary reflection models.

5. References

5.1. Books

[1]: Brian C. J. Moore “An introduction to the Psychology of Hearing”, 5th edition, 2004, Elsevier Academic Press.

[2]: W: Marshall Leach, Jr. “Introduction to Electroacoustics & Audio Amplifier Design”, 3rd edition, 2003, Kendall/Hunt Publishing Company.

[3]: Leo L. Beranek, “Acoustics”, Acoustical Society of America, 1993 edition, 1996.

[4] Finn Jacobsen, Peter Juhl “Radiation of Sound”, not published, DTU, 2005.

[5] Hans Ebert “Elektronik Ståbi”, Teknisk Forlag, 1995.

[6] http://ccrma.stanford.edu/~jos/filters/Group_Delay.html.

5.2. Papers

Paper: S. Linkwitz, 'Active Crossover Networks for Non-Coincident Drivers,' *J. Audio Eng. Soc.,* vol. 24, pp. 2-8 (Jan./Feb. 1976) – together with his co-worker Russ Riley

5.3. Links

http://sound.westhost.com/lr-passive.htm - A very nice introduction to the design of a crossover network, including compensation of loudspeaker impedance, voice coil temperature and much more.

6. Appendix

6.1. *Plot transfer function*

Amplitude and phase responses are plotted using the following MATLAB shell script. Representative functions addressed within the script are included in the following section.

6.1.1. *Main script*

The file was executed by entering *plot_transfer_function* at the MATLAB prompt.

```
% Plot transfer functions.

clear all                   % Remove old stuff.

% -------------------------------------------------------------------------
% Constants.
% -------------------------------------------------------------------------

% Frequency.
    FBEG  =      20;        % Start frequency (Hz).
    FEND  =   20000;        % Stop frequency (Hz).
    FSTEP =       1;        % Frequency increment (Hz).
    f = [FBEG:FSTEP:FEND];  % Frequency f.
    g = f + FSTEP;          % Frequency g (required for group delay).

% Acoustical.
    F0  =  4000;            % Crossover frequency (Hz).
    c   =   343;            % Speed of sound (m/s).
    L   =     0.25;         % Displacement between loudspeakers (m).

% Miscellaneous.
    D2R = pi/180;           % Conversion from degrees to radians.
    R2D = 180/pi;           % Conversion from radians to degrees.
    TRUE  = logical(1);     % Used by the IF statement.
    FALSE = logical(0);

% -------------------------------------------------------------------------
% Crossover network coefficients.
% -------------------------------------------------------------------------

% Second order, N = 2:
    A1 = 2.00;          % 1.00

% Third order, N = 3:
%   A1 = 3.00;
%   A2 = 3.00;

% Fourth order, N = 4:
%   A1 = 2.83;          % 2.00, 2.61
%   A2 = 4.00;          % 3.00, 3.41
%   A3 = 2.83;          % 2.00, 2.61

% Sixth order, N = 6:
%   A1 = 1.00;
%   A2 = 3.00;
%   A3 = 6.00;
%   A4 = 3.00;
%   A5 = 1.00;

% -------------------------------------------------------------------------
% Crossover network at frequency f.
% -------------------------------------------------------------------------

    bass_f     = 1;
%   treble_f   = 1;
```

```
%   bass_f     = lowpass_1(F0,f);
%   treble_f   = highpass_1(F0,f);
%   bass_f     = lowpass_2(F0,A1,f);
%   midrange_f = bandpass_2(F0,A1,f);
    treble_f   = highpass_2(F0,A1,f);
%   bass_f     = lowpass_3(F0,A1,A2,f);
%   midrange_f = bandpass_3(F0,A1,A2,f);
%   treble_f   = highpass_3(F0,A1,A2,f);
%   bass_f     = lowpass_4(F0,A1,A2,A3,f);
%   midrange_f = bandpass_4(F0,A1,A2,A3,f);
%   treble_f   = highpass_4(F0,A1,A2,A3,f);
%   bass_f     = lowpass_6(F0,A1,A2,A3,A4,A5,f);
%   treble_f   = highpass_6(F0,A1,A2,A3,A4,A5,f);

% ------------------------------------------------------------------------
% Crossover network at frequency g.
% ------------------------------------------------------------------------

    bass_g     = 1;
%   treble_g   = 1;
%   bass_g     = lowpass_1(F0,g);
%   treble_g   = highpass_1(F0,g);
%   bass_g     = lowpass_2(F0,A1,g);
%   midrange_g = bandpass_2(F0,A1,g);
    treble_g   = highpass_2(F0,A1,g);
%   bass_g     = lowpass_3(F0,A1,A2,g);
%   midrange_g = bandpass_3(F0,A1,A2,g);
%   treble_g   = highpass_3(F0,A1,A2,g);
%   bass_g     = lowpass_4(F0,A1,A2,A3,g);
%   midrange_g = bandpass_4(F0,A1,A2,A3,g);
%   treble_g   = highpass_4(F0,A1,A2,A3,g);
%   bass_g     = lowpass_6(F0,A1,A2,A3,A4,A5,g);
%   treble_g   = highpass_6(F0,A1,A2,A3,A4,A5,g);

% ------------------------------------------------------------------------
% Include loudspeaker models.
% ------------------------------------------------------------------------

% Treble loudspeaker.
    FCT  = 20e3;        % Voice coil cutoff pole frequency (Hz).
    FLT  = 10e6;        % Voice coil cutoff null frequency (Hz).
    FST  =  1e3;        % Mechanical resonance frequency (Hz).
    QTCT =    1;        % Total quality factor.

    treble_f = loudspeaker(FCT,FLT,FST,QTCT,treble_f,f);
    treble_g = loudspeaker(FCT,FLT,FST,QTCT,treble_g,g);

% Midrange loudspeaker.
    FCM  = 10e3;        % Voice coil cutoff pole frequency (Hz).
    FLM  = 10e6;        % Voice coil cutoff null frequency (Hz).
    FSM  =  100;        % Mechanical resonance frequency (Hz).
    QTCM =    1;        % Total quality factor.

%   midrange_f = loudspeaker(FCM,FLM,FSM,QTCM,midrange_f,f);
%   midrange_g = loudspeaker(FCM,FLM,FSM,QTCM,midrange_g,g);

% Bass loudspeaker.
    FCB  =  3e3;        % Voice coil cutoff pole frequency (Hz).
    FLB  = 10e6;        % Voice coil cutoff null frequency (Hz).
    FSB  =   70;        % Mechanical resonance frequency (Hz).
    QTCB =    0.7;      % Total quality factor.

    bass_f = loudspeaker(FCB,FLB,FSB,QTCB,bass_f,f);
    bass_g = loudspeaker(FCB,FLB,FSB,QTCB,bass_g,g);

% ------------------------------------------------------------------------
% Introduce directivity.
% ------------------------------------------------------------------------

    THETA =   0;        % Horisontal or vertical ongle (degrees).
    AT =  10e-3;        % Treble diaphragm radius (m).
    AM =  50e-3;        % Midrange diaphragm radius (m).
    AB = 100e-3;        % Bass diaphragm radius (m).
```

```
%   treble_f   = directivity(THETA,AT,f).*treble_f;
%   treble_g   = directivity(THETA,AT,g).*treble_g;
%   midrange_f = directivity(THETA,AM,f).*midrange_f;
%   midrange_g = directivity(THETA,AM,g).*midrange_g;
%   bass_f     = directivity(THETA,AB,f).*bass_f;
%   bass_g     = directivity(THETA,AB,g).*bass_g;

% ---------------------------------------------------------------------------
% Introduce phase due to vertical angle.
% ---------------------------------------------------------------------------

% Vertical offset angle.
    VA =   0;           % Vertical offset angle (degrees).

%   Two-loudspeaker arrangement (bass + treble).
%   treble_f   = exp( i*2*pi*f*L*sin(VA*D2R)/(2*c)).*treble_f;
%   treble_g   = exp( i*2*pi*g*L*sin(VA*D2R)/(2*c)).*treble_g;
%   bass_f     = exp(-i*2*pi*f*L*sin(VA*D2R)/(2*c)).*bass_f;
%   bass_g     = exp(-i*2*pi*g*L*sin(VA*D2R)/(2*c)).*bass_g;
%   midrange_f = midrange_f;
%   midrange_g = midrange_g;

%   Three-loudspeaker arrangement (bass + treble + bass).
%   treble_f  = treble_f;
%   treble_g  = treble_g;
%   bass_f    = (exp( i*2*pi*f*L*sin(VA*D2R)/c) + ...
%                exp(-i*2*pi*f*L*sin(VA*D2R)/c)).*bass_f/2;
%   bass_g    = (exp( i*2*pi*g*L*sin(VA*D2R)/c) + ...
%                exp(-i*2*pi*g*L*sin(VA*D2R)/c)).*bass_g/2;

% ---------------------------------------------------------------------------
% Resultant transfer functions.
% ---------------------------------------------------------------------------

    ATT = 1.0;          % Attenuator network.
    DLY = 0e-6;         % Delay (s).

%   sum_f = bass_f + treble_f;                 % Two-way.
%   sum_g = bass_g + treble_g;

    sum_f = bass_f - ATT*treble_f;             % Two-way, inverted.
    sum_g = bass_g - ATT*treble_g;

%   sum_f = bass_f - treble_f.*exp(-i*f*DLY);   % Two-way, time shifted.
%   sum_g = bass_g - treble_g.*exp(-i*f*DLY);

%   sum_f = bass_f + midrange_f + treble_f;     % Three-way.
%   sum_g = bass_g + midrange_g + treble_g;

%   sum_f = ones(size(f));                     % Dummy (unity sums).
%   sum_g = ones(size(g));

% ---------------------------------------------------------------------------
% Include diffraction model.
% ---------------------------------------------------------------------------

    sum_f = diffraction_sectional(0.18,0.22,0.22,0.27, f).*sum_f;
    sum_g = diffraction_sectional(0.18,0.22,0.22,0.27, f).*sum_g;

% ---------------------------------------------------------------------------
% Include one-dimensional boundary reflection model.
% ---------------------------------------------------------------------------

    H1 = 0.25;          % Loudspeaker distance above floor (m).
    H0 = 1.00;          % Listener distance above floor (m).
    L  = 2.00;          % Horizontal distance to listener (m).

    RD = sqrt((H0 - H1)^2 + L^2);
    RR = sqrt((H0 + H1)^2 + L^2);
    TH = R2D*(atan((H0 - H1)/L) + atan((H0 + H1)/L));
    sum_f = boundary_simple(RD,RR,TH,AB,f).*sum_f;

% ---------------------------------------------------------------------------
% Plot amplitude spectrum.
```

```
% -------------------------------------------------------------------------

figure(1);
semilogx(f, 20*log10(abs(sum_f)), ...
         f, 20*log10(abs(treble_f)), ...
         f, 20*log10(abs(bass_f)));
legend('Total','Treble','Bass');
axis([FBEG FEND -20 5]);
ylabel('Amplitude (dB)');
xlabel('Frequency (Hz)');

% -------------------------------------------------------------------------
% Plot phase spectrum.
% -------------------------------------------------------------------------

figure(2);
semilogx(f, R2D*angle(sum_f), ...
         f, R2D*angle(treble_f), ...
         f, R2D*angle(bass_f));
legend('Total','Treble','Bass');
axis([FBEG FEND -200 200]);
ylabel('Phase (°)');
xlabel('Frequency (Hz)');

% -------------------------------------------------------------------------
% Plot group delay spectrum.
% -------------------------------------------------------------------------

% Group delay = -Im(H'/H).

%   groupdelay = -imag((sum_g - sum_f)./(sum_f*FSTEP));
%   figure(3);
%   semilogx(f, groupdelay);
%   axis([FBEG FEND 0 1e-3]);
%   ylabel('Group delay (s)');
%   xlabel('Frequency (Hz)');
```

6.1.2. Filter function

All filters, being low-pass, band-pass or high-pass, are written in the below style. Input "f0" is the centre frequency of the filter and A1 and so forth are coefficients to the polynomials and are specified from the main script file. Input "f" is a frequency vector, either from 0.1 to 10 in steps of 0.01 (normalised frequency) or from 20 Hz to 20000 Hz in steps of 1 Hz. Different filter flavours are specified for several filters but only one is enabled by un-commenting the relevant definition.

```
% Second order low-pass filter.
function out = lowpass_2(f0,A1, f);
s0 = (i/f0)*f;
% out = (1 + A1*s0)./(1 + A1*s0 + s0.^2);
% out = (1 + (A1/2)*s0)./(1 + A1*s0 + s0.^2);
out = 1./(1 + A1*s0 + s0.^2);
```

Only the last equation is enabled.

6.1.3. Loudspeaker

A loudspeaker is defined by a second-order high-pass filter and a first-order low-pass filter. Input coefficients are specified from the main script. Variable *in* is the input response, which is returned with the loudspeaker transfer function.

```
% Loudspeaker transfer function.

function out = loudspeaker(FC, FL, FS, QTC, in, f);

% FC  = Voice coil pole frequency due to RE and LE (Hz).
% FL  = Voice coil null frequency due to eddy currents (Hz).
% FS  = Mechanical resonance frequency (Hz).
% QTC = Total quality factor.

out = ((1+i*f/FL)./(1+i*f/FC)) .* ...
      ( ((i*f/FS).^2)./(1+i*f/(FS*QTC)+(i*f/FS).^2) ).*in;
```

6.1.4. Directivity

Directivity is defined by the Bessel-function *besselj*, which is of the first kind. The if-statement avoids division by zero at low frequencies and zero angle.

```
function out = directivity(T, A, f)

% T  = Angle (degrees).
% A  = Loudspeaker radius (m).
% f  = Frequency (Hz).

TT = (pi/180)*T;        % Convert angle from degrees to radians.
ka = 2*pi*f*A/343;      % Angular wave number.
out = ones(size(f));

if (abs(TT) > eps)
    out = abs(2*besselj(1,ka*sin(TT))./(ka*sin(TT)));
end
```

6.1.5. Diffraction

A simplified model is used, which is based upon the circular section method, here limited to four sections (90° each).

```
function out = diffraction_sectional(R1,R2,R3,R4, f)

% R1 = Radius of section 1 (m).
% R2 = Radius of section 2 (m).
% R3 = Radius of section 3 (m).
% R4 = Radius of section 4 (m).

ikR1 = i*2*pi*f*R1/343;
ikR2 = i*2*pi*f*R2/343;
ikR3 = i*2*pi*f*R3/343;
ikR4 = i*2*pi*f*R4/343;

out = 1-( exp(-ikR1)+exp(-ikR2)+exp(-ikR3)+exp(-ikR4) )/8;
```

6.1.6. Boundary reflections

A simplified model is used for reflection from the boundary, which is only using one reflecting surface. The model includes loudspeaker directivity and assumes that the loudspeaker is pointing directly toward the listener.

```
function out = boundary_simple(RD,RR,TH,A,f);

% PD = Direct path length (m).
```

```
% PR = Reflected path length (m).

ik = i*2*pi*f/343;

out = 1 + (RD/RR)*directivity(TH,A,f).*exp(-ik*(RR-RD));
```

6.2. Plot boundary reflection

This is the full script for calculation of the effect of boundary reflections. The model assume a rectangular room with six reflecting surfaces. The initial part of the script checks for negative or too large coordinates and zero diaphragm diameter.

```
% Compute the resultant sound pressure for a loudspeaker within a room.

clear all;

% --------------------------------------------------------------------------
% Input parameters.
% --------------------------------------------------------------------------

% Room dimensions (m):
XL  = 50;       YL  = 30;       ZL  = 15;

% Observation point (m):
X0  = 40;       Y0  =  10;      Z0  =  3;

%Loudspeaker 1 (m):
X1  =  5;       Y1  =   5;     Z1  =  10;

A1H =  0;   % Loudspeaker horisontal angle (degrees).
A1V =  0;   % Loudspeaker vertical angle (degrees).

C1  =  0.3; % Reflection coefficient for wall behind loudspeaker (m).
C2  =  0.3; % Reflection coefficient for left wall (m).
C3  =  0.1; % Reflection coefficient for floor (m).
C4  =  0.1; % Reflection coefficient for wall behind listener (m).
C5  =  0.3; % Reflection coefficient for right wall (m).
C6  =  0.1; % Reflection coefficient for roof (m).

A =    0.1; % Loudspeaker diaphragm radius (m).

f = [10:1:10000];       % Frequency range.

% --------------------------------------------------------------------------
% Check consistency of input parameters.
% --------------------------------------------------------------------------

if (min([X0 Y0 Z0 X1 Y1 Z1]) < 0)
    error('Coordinates (x,y,z) must not be negative.');
end
if (min([XL YL ZL]) < 0)
    error('Room coordinates must not be negative.');
end
if (min([(XL-X0) (YL-Y0) (ZL-Z0) (XL-X1) (YL-Y1) (ZL-Z1)]) < 0)
    error('oordinates (x,y,z) must not exceed room limits.');
end
if (min([C1 C2 C3 C4 C5 C6]) < 0)
    error('Reflection coefficients (C) must not be negative.');
end
if (max([C1 C2 C3 C4 C5 C6]) > 1)
    error('Reflection coefficients (C) must be maximum unity.');
end
if (A <= 0)
    error('Loudspeaker diaphragm diameter must be positive.');
end

% --------------------------------------------------------------------------
% Calculate constant vectors.
```

```
% ------------------------------------------------------------------------

% Distance vectors for loudspeaker 1:
P1D  = [(X0-X1)         (Y0-Y1)        (Z0-Z1)];         % Direct signal.
P1R1 = [(X0+X1)         (Y0-Y1)        (Z0-Z1)];         % Reflection 1.
P1R2 = [(X0-X1)         (Y0+Y1)        (Z0-Z1)];         % Reflection 2.
P1R3 = [(X0-X1)         (Y0-Y1)        (Z0+Z1)];         % Reflection 3.
P1R4 = [(X0-(2*XL-X1))  (Y0-Y1)        (Z0-Z1)];         % Reflection 4.
P1R5 = [(X0-X1)         (Y0-(2*YL-Y1)) (Z0-Z1)];         % Reflection 5.
P1R6 = [(X0-X1)         (Y0-Y1)        (Z0-(2*ZL-Z1))];  % Reflection 6.

% Loudspeaker direction vector:
L1X = cos(pi*A1H/180)*cos(pi*A1V/180);      % Coordinate x.
L1Y = sin(pi*A1H/180)*cos(pi*A1V/180);      % Coordinate y.
L1Z = sin(pi*A1V/180);                      % Coordinate z.

L1D  = [ L1X  L1Y  L1Z];
L1R1 = [-L1X  L1Y  L1Z];
L1R2 = [ L1X -L1Y  L1Z];
L1R3 = [ L1X  L1Y -L1Z];
L1R4 = [-L1X  L1Y  L1Z];
L1R5 = [ L1X -L1Y  L1Z];
L1R6 = [ L1X  L1Y -L1Z];

% ------------------------------------------------------------------------
% Observation angles:
% ------------------------------------------------------------------------

T1   = acos((L1D *P1D') /(norm(L1D) *norm(P1D)));         % Direct signal.
T1R1 = acos((L1R1*P1R1')/(norm(L1R1)*norm(P1R1)));        % Reflection 1.
T1R2 = acos((L1R2*P1R2')/(norm(L1R2)*norm(P1R2)));        % Reflection 2.
T1R3 = acos((L1R3*P1R3')/(norm(L1R3)*norm(P1R3)));        % Reflection 3.
T1R4 = acos((L1R4*P1R4')/(norm(L1R4)*norm(P1R4)));        % Reflection 4.
T1R5 = acos((L1R5*P1R5')/(norm(L1R5)*norm(P1R5)));        % Reflection 5.
T1R6 = acos((L1R6*P1R6')/(norm(L1R6)*norm(P1R6)));        % Reflection 6.

% Check that the angls are non-zero (>2.2e-16) to avoid division by zero.
if (abs(T1)<eps)   T1   = eps; end
if (abs(T1R1)<eps) T1R1 = eps; end
if (abs(T1R2)<eps) T1R2 = eps; end
if (abs(T1R3)<eps) T1R3 = eps; end
if (abs(T1R4)<eps) T1R4 = eps; end
if (abs(T1R5)<eps) T1R5 = eps; end
if (abs(T1R6)<eps) T1R6 = eps; end

% ------------------------------------------------------------------------
% Directivity.
% ------------------------------------------------------------------------

k = (2*pi/343).*f;      % Expand k vector.

% Directivities:
D1D=   2*besselj(1,k*A*sin(T1))  ./(k*A*sin(T1));       % Direct signal.
D1R1 = 2*besselj(1,k*A*sin(T1R1))./(k*A*sin(T1R1));     % Reflection 1.
D1R2 = 2*besselj(1,k*A*sin(T1R2))./(k*A*sin(T1R2));     % Reflection 2.
D1R3 = 2*besselj(1,k*A*sin(T1R3))./(k*A*sin(T1R3));     % Reflection 3.
D1R4 = 2*besselj(1,k*A*sin(T1R4))./(k*A*sin(T1R4));     % Reflection 4.
D1R5 = 2*besselj(1,k*A*sin(T1R5))./(k*A*sin(T1R5));     % Reflection 5.
D1R6 = 2*besselj(1,k*A*sin(T1R6))./(k*A*sin(T1R6));     % Reflection 6.

% ------------------------------------------------------------------------
% Sound pressure.
% ------------------------------------------------------------------------

sp = norm(P1D)*(D1D/norm(P1D)      .*exp(-i*k*norm(P1D))  + ...
               (C1*D1R1/norm(P1R1)).*exp(-i*k*norm(P1R1)) + ...
               (C2*D1R2/norm(P1R2)).*exp(-i*k*norm(P1R2)) + ...
               (C3*D1R3/norm(P1R3)).*exp(-i*k*norm(P1R3)) + ...
               (C4*D1R4/norm(P1R4)).*exp(-i*k*norm(P1R4)) + ...
               (C5*D1R5/norm(P1R5)).*exp(-i*k*norm(P1R5)) + ...
               (C6*D1R6/norm(P1R6)).*exp(-i*k*norm(P1R6)) );

% ------------------------------------------------------------------------
% Plot data.
```

```
% -------------------------------------------------------------------------

semilogx(f, 20.*log10(abs(sp)));
ylabel('Sound pressure level (dB)');
xlabel('Frequency (Hz)');
axis([10 10000 -20 15]);
```